

H^∞ FUNCTIONAL CALCULUS AND SQUARE FUNCTION ESTIMATES FOR RITT OPERATORS

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ABSTRACT. A Ritt operator $T: X \rightarrow X$ on Banach space is a power bounded operator satisfying an estimate $n\|T^n - T^{n-1}\| \leq C$. When $X = L^p(\Omega)$ for some $1 < p < \infty$, we study the validity of square functions estimates $\|(\sum_k k|T^k(x) - T^{k-1}(x)|^2)^{\frac{1}{2}}\|_{L^p} \lesssim \|x\|_{L^p}$ for such operators. We show that T and T^* both satisfy such estimates if and only if T admits a bounded functional calculus with respect to a Stolz domain. This is a single operator analog of the famous Cowling-Doust-McIntosh-Yagi characterization of bounded H^∞ -calculus on L^p -spaces by the boundedness of certain Littlewood-Paley-Stein square functions. We also prove a similar result on Hilbert space. Then we extend the above to more general Banach spaces, where square functions have to be defined in terms of certain Rademacher averages. We focus on noncommutative L^p -spaces, where square functions are quite explicit, and we give applications, examples and illustrations on those spaces, as well as on classical L^p .

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1. INTRODUCTION

Let X be a Banach space and let $T: X \rightarrow X$ be a bounded operator. If $F \subset \mathbb{C}$ is any compact set containing the spectrum of T , a natural question is whether there is an estimate

$$(1.1) \quad \|\varphi(T)\| \leq K \sup\{|\varphi(\lambda)| : \lambda \in F\}$$

satisfied by all rational functions φ . The mapping $\varphi \mapsto \varphi(T)$ on rational functions is the most elementary form of a ‘holomorphic functional calculus’ associated to T and (1.1) means that this functional calculus is bounded in an appropriate sense.

The most famous such functional calculus estimate is von Neumann’s inequality, which says that if $F = \overline{\mathbb{D}}$ is the closed unit disc centered at 0, then (1.1) holds true with $K = 1$ for any contraction T on Hilbert space. Von Neumann’s inequality was a source of inspiration for the development of various topics around functional calculus estimates on Hilbert space, including polynomial boundedness, K -spectral sets and related similarity problems. We refer the reader to [5, 42, 43, 46] and the references therein for a large information. See also [13, 11] for striking results in the case when F is equal to the numerical range of T .

When X is a non Hilbertian Banach space, our knowledge on operators $T: X \rightarrow X$ and compact sets F satisfying (1.1) for some $K \geq 1$ is quite limited. Positive examples are provided by scalar type operators (see [14]). A more significant observation is that this issue is closely related to H^∞ -functional calculus associated to sectorial operators and indeed, that

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topic will play a key role in this paper. H^∞ -functional calculus was introduced by McIntosh and his co-authors in [10, 39] and was then developed and applied successfully to various areas, in particular to the study of maximal regularity for certain PDE's, to harmonic analysis of semigroups, and to multiplier theory. We refer the reader to [27] for relevant information.

In this paper we deal with holomorphic functional calculus for Ritt operators. Recall that by definition, $T: X \rightarrow X$ is a Ritt operator provided that T is power bounded and there exists a constant $C > 0$ such that $n\|T^n - T^{n-1}\| \leq C$ for any integer $n \geq 1$. In this case, the spectrum of T is included in the closure $\overline{B_\gamma}$ of a Stolz domain of the unit disc, see Section 2 and Figure 1 below for details. In accordance with the preceding discussion, this leads to the question whether T satisfies an estimate (1.1) for $F = \overline{B_\gamma}$. We will say that T has a bounded $H^\infty(B_\gamma)$ functional calculus in this case (this terminology will be justified in Section 2). The general problem motivating the present work is to characterize Ritt operators having a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$ and to exhibit explicit classes of operators satisfying this property.

If $X = H$ is a Hilbert space and $T: H \rightarrow H$ is a bounded operator, we define the ‘square function’

$$(1.2) \quad \|x\|_T = \left(\sum_{k=1}^{\infty} k \|T^k(x) - T^{k-1}(x)\|_H^2 \right)^{\frac{1}{2}}, \quad x \in H.$$

Likewise for any measure space (Ω, μ) , for any $1 \leq p < \infty$ and for any $T: L^p(\Omega) \rightarrow L^p(\Omega)$, we consider

$$(1.3) \quad \|x\|_T = \left\| \left(\sum_{k=1}^{\infty} k |T^k(x) - T^{k-1}(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}, \quad x \in L^p(\Omega).$$

Let $T: X \rightarrow X$ be a Ritt operator on either $X = H$ or $X = L^p(\Omega)$. It was implicitly proved in [33] that if T has a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$, then it satisfies a uniform estimate $\|x\|_T \lesssim \|x\|$.

This paper has two main purposes. First we establish a converse to this result and prove the following. (Here $p' = \frac{p}{p-1}$ is the conjugate number of p .)

Theorem 1.1. *Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a Ritt operator, with $1 < p < \infty$. The following assertions are equivalent.*

- (i) *The operator T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*
- (ii) *The operator T and its adjoint $T^*: L^{p'}(\Omega) \rightarrow L^{p'}(\Omega)$ both satisfy uniform estimates*

$$\|x\|_T \lesssim \|x\|_{L^p} \quad \text{and} \quad \|y\|_{T^*} \lesssim \|y\|_{L^{p'}}$$

for $x \in L^p(\Omega)$ and $y \in L^{p'}(\Omega)$.

We also prove a similar result for Ritt operators on Hilbert space.

Second, we investigate relationships between the existence of a bounded $H^\infty(B_\gamma)$ functional calculus and adapted square function estimates on general Banach spaces. We pay a special attention to noncommutative L^p -spaces and prove square function estimates for large classes of Schur multipliers and selfadjoint Markov operators on those spaces.

Ritt operators can be considered as discrete analogs of sectorial operators of type $< \frac{\pi}{2}$, as explained e.g. in [7, 8] or [33, Section 2]. According to this analogy, Theorem 1.1 and its Hilbertian counterpart should be regarded as discrete analogs of the main results of [10, 39] showing the equivalence between the boundedness of H^∞ -functional calculus and some square function estimates for sectorial operators. Likewise, in the noncommutative setting, our results are both an analog and an extension of the main results of the memoir [21].

The definitions of the discrete square functions (1.2) and (1.3) go back at least to [51], where they were used to study selfadjoint Markov operators and diffusion semigroups on classical (=commutative) L^p -spaces. They appeared in the context of Ritt operators in [23] and [33, 34].

We now turn to a brief description of the paper. In Sections 2 and 3, we introduce $H^\infty(B_\gamma)$ functional calculus and square functions for Ritt operators, and we prove basic preliminary results. Our definition of square functions on general Banach spaces relies on Rademacher averages. Regarding such averages as abstract square functions is a well-known principle, see e.g. [48, 25, 21] for illustrations. If T is a Ritt operator, then $A = I_X - T$ is a sectorial operator and we show in Section 4 that T has a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{\pi}{2}$ if and only if A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{\pi}{2}$. This observation, stated as Proposition 4.1, provides a tool to transfer bounded H^∞ -calculus results from the sectorial setting to Ritt operators. There is apparently no similar way to compare square functions associated to T to square functions associated to A . This is at the root of most of the difficulties in our analysis of Ritt operators. Proposition 4.1 will be applied in Section 8, where we give applications and illustrations on Hilbert spaces, classical L^p -spaces, and noncommutative L^p -spaces.

We will make use of R -boundedness and the notion of R -Ritt operators. That class was introduced by Blunck [7, 8] as a discrete counterpart of R -sectorial operators. Our first main result, proved in Section 5, says that if a Ritt operator $T: L^p(\Omega) \rightarrow L^p(\Omega)$ satisfies condition (ii) in Theorem 1.1 above, then it is actually an R -Ritt operator. In Section 6, we show that for a large class of Banach spaces X , including spaces with property (α) and noncommutative L^p -spaces, any Ritt operator $T: X \rightarrow X$ with a bounded $H^\infty(B_\gamma)$ functional calculus satisfies square function estimates. This is based on the study of a strong form of H^∞ -functional calculus called ‘quadratic H^∞ -functional calculus’, where scalar valued holomorphic functions are replaced by ℓ^2 -valued ones. Section 7 is devoted to the converse problem of whether square function estimates for T and T^* imply a bounded $H^\infty(B_\gamma)$ functional calculus. We show that this holds true whenever T is R -Ritt, and complete the proofs of Theorem 1.1 and similar equivalence results.

We finally give a few notation to be used along this paper. We let $B(X)$ denote the algebra of all bounded operators on X and we let I_X denote the identity operator on X (or simply I if there is no ambiguity on X). We let $\sigma(T)$ denote the spectrum of an operator T (bounded or not) and we let $R(\lambda, T) = (\lambda I_X - T)^{-1}$ denote the resolvent operator when λ belongs to the resolvent set $\mathbb{C} \setminus \sigma(T)$. Next, we let $\text{Ran}(T)$ and $\text{Ker}(T)$ denote the range and the kernel of T , respectively.

For any $a \in \mathbb{C}$ and $r > 0$, we let $D(a, r)$ denote the open disc of radius r centered at a . Also, we let $\mathbb{D} = D(0, 1)$ denote the open unit disc. For any non empty open set $\emptyset \subset \mathbb{C}$ and any Banach space Z , we let $H^\infty(\emptyset; Z)$ denote the space of all bounded holomorphic functions $\varphi: \emptyset \rightarrow Z$. This is a Banach space for the supremum norm

$$\|\varphi\|_{H^\infty(\emptyset; Z)} = \sup\{\|\varphi(\lambda)\|_Z : \lambda \in \emptyset\}.$$

In the scalar case, we write $H^\infty(\emptyset)$ instead of $H^\infty(\emptyset; \mathbb{C})$ and $\|\varphi\|_{\infty, \emptyset}$ instead of $\|\varphi\|_{H^\infty(\emptyset)}$. Finally we let \mathcal{P} denote the algebra of complex polynomials.

In Theorem 1.1 and later on in the paper we use the notation \lesssim to indicate an inequality up to a constant which does not depend on the particular element to which it applies. Then $A(x) \approx B(x)$ will mean that we both have $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

2. RITT OPERATORS AND THEIR FUNCTIONAL CALCULUS

We start this section with some classical background on the H^∞ -functional calculus associated to sectorial operators. The construction and basic properties below go back to [10, 39], see also [24, 29] for complements.

For any $\omega \in (0, \pi)$, we let

$$(2.1) \quad \Sigma_\omega = \{z \in \mathbb{C}^* : |\operatorname{Arg}(z)| < \omega\}$$

be the open sector of angle 2ω around the positive real axis $(0, \infty)$.

Let X be a Banach space. We say that a closed linear operator $A: D(A) \rightarrow X$ with dense domain $D(A) \subset X$ is sectorial of type ω if $\sigma(A) \subset \overline{\Sigma_\omega}$ and for any $\nu \in (\omega, \pi)$, the set

$$(2.2) \quad \{zR(z, A) : z \in \mathbb{C} \setminus \overline{\Sigma_\nu}\}$$

is bounded.

For any $\theta \in (0, \pi)$, let $H_0^\infty(\Sigma_\theta)$ denote the algebra of all bounded holomorphic functions $f: \Sigma_\theta \rightarrow \mathbb{C}$ for which there exists two positive real numbers $s, c > 0$ such that

$$|f(z)| \leq c \frac{|z|^s}{1 + |z|^{2s}}, \quad z \in \Sigma_\theta.$$

Let $0 < \omega < \theta < \pi$ and let $f \in H_0^\infty(\Sigma_\theta)$. Then we set

$$(2.3) \quad f(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} f(z) R(z, A) dz,$$

where $\nu \in (\omega, \theta)$ and the boundary $\partial \Sigma_\nu$ is oriented counterclockwise. The sectoriality condition ensures that this integral is absolutely convergent and defines an element of $B(X)$. Moreover by Cauchy's Theorem, this definition does not depend on the choice of ν . Further the resulting mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$ which is consistent with the usual functional calculus for rational functions.

We say that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if the latter homomorphism is bounded, that is, there exists a constant $K > 0$ such that

$$\|f(A)\| \leq K \|f\|_{\infty, \Sigma_\theta}, \quad f \in H_0^\infty(\Sigma_\theta).$$

If A has a dense range and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, then the above homomorphism naturally extends to a bounded homomorphism $f \mapsto f(A)$ from the whole space $H^\infty(\Sigma_\theta)$ into $B(X)$.

It is well-known that the above construction can be adapted to various contexts, see e.g. [19] and [15]. We shall briefly explain below such a functional calculus construction for Ritt operators. We first recall some background on this class.

We say that an operator $T: X \rightarrow X$ is a Ritt operator provided that the two sets

$$(2.4) \quad \{T^n : n \geq 0\} \quad \text{and} \quad \{n(T^n - T^{n-1}) : n \geq 1\}$$

are bounded. The following spectral characterization is crucial: T is a Ritt operator if and only if

$$\sigma(T) \subset \overline{\mathbb{D}} \quad \text{and} \quad \{(\lambda - 1)R(\lambda, T) : |\lambda| > 1\} \quad \text{is bounded.}$$

Indeed this condition is often taken as a definition for Ritt operators. We refer to [38, 40] for this characterization and also to [41], which contains the key argument, and to [7, 8] and [33, Section 2] for complements. Let

$$A = I - T.$$

It follows from the above referred papers that T is a Ritt operator if and only if

$$(2.5) \quad \sigma(T) \subset \mathbb{D} \cup \{1\} \quad \text{and} \quad A \text{ is a sectorial operator of type } < \frac{\pi}{2}.$$

We will need quantitative versions of the above equivalence property. For that purpose we introduce the Stolz domains B_γ as on Figure 1 below. Namely for any angle $\gamma \in (0, \frac{\pi}{2})$, we let B_γ be the interior of the convex hull of 1 and the disc $D(0, \sin \gamma)$.

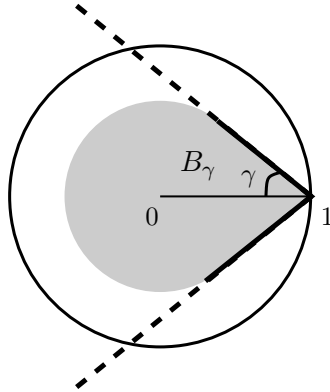


FIGURE 1.

Lemma 2.1. *An operator $T: X \rightarrow X$ is a Ritt operator if and only if there exists an angle $\alpha \in (0, \frac{\pi}{2})$ such that*

$$(2.6) \quad \sigma(T) \subset \overline{B_\alpha}$$

and for any $\beta \in (\alpha, \frac{\pi}{2})$, the set

$$(2.7) \quad \{(\lambda - 1)R(\lambda, T) : \lambda \in \mathbb{C} \setminus \overline{B_\beta}\}$$

is bounded.

In this case, $A = I - T$ is a sectorial operator of type α .

Proof. Assume that T is a Ritt operator and let us apply (2.5). Let $\omega \in (0, \frac{\pi}{2})$ be a sectorial type of A . Then $\sigma(T)$ is both included in $\mathbb{D} \cup \{1\}$ and in the cone $1 - \overline{\Sigma_\omega}$ hence there exists $\omega \leq \alpha < \frac{\pi}{2}$ such that $\sigma(T) \subset \overline{B_\alpha}$.

Consider the function h on $\mathbb{C} \setminus \sigma(T)$ defined by $h(\lambda) = (\lambda - 1)R(\lambda, T)$. This function is bounded on $\mathbb{C} \setminus \overline{D}(0, 2)$. Indeed if we let $C_0 = \sup_{n \geq 0} \|T^n\|$, then writing

$$R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

when $|\lambda| > 1$, we have $\|R(\lambda, T)\| \leq C_0/(|\lambda| - 1)$, and hence

$$|\lambda - 1| \|R(\lambda, T)\| \leq C_0 \frac{|\lambda| + 1}{|\lambda| - 1}, \quad \lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

Let $\beta \in (\alpha, \frac{\pi}{2})$. The compact set

$$(2.8) \quad \Lambda_\beta = \{\lambda \in 1 - \overline{\Sigma_\beta} : \operatorname{Re}(\lambda) \leq \sin^2 \beta \text{ and } \sin \beta \leq |\lambda| \leq 2\}$$

is included in the resolvent set of T , hence h is bounded on Λ_β . Furthermore

$$h(\lambda) = (1 - \lambda)R((1 - \lambda), A)$$

and A is sectorial of type α . Consequently h is bounded outside $1 - \overline{\Sigma_\beta}$. Altogether, this shows that h is bounded outside $\overline{B_\beta}$.

The rest of the statement is obvious. \square

The above lemma leads to the following.

Definition 2.2. We say that $T: X \rightarrow X$ is a Ritt operator of type $\alpha \in (0, \frac{\pi}{2})$ if it satisfies the conclusions of Lemma 2.1.

Then we construct an H^∞ -functional calculus as follows. For any $\gamma \in (0, \frac{\pi}{2})$, we let $H_0^\infty(B_\gamma) \subset H^\infty(B_\gamma)$ be the space of all bounded holomorphic functions $\varphi: B_\gamma \rightarrow \mathbb{C}$ for which there exists two positive real numbers $s, c > 0$ such that

$$(2.9) \quad |\varphi(\lambda)| \leq c|1 - \lambda|^s, \quad \lambda \in B_\gamma.$$

Assume that T has type α and $\gamma \in (\alpha, \frac{\pi}{2})$. Then for any $\varphi \in H_0^\infty(B_\gamma)$, we define

$$(2.10) \quad \varphi(T) = \frac{1}{2\pi i} \int_{\partial B_\beta} \varphi(\lambda) R(\lambda, T) d\lambda,$$

where $\beta \in (\alpha, \gamma)$ and the boundary ∂B_β is oriented counterclockwise. The boundedness of $\{(\lambda - 1)R(\lambda, T) : \lambda \in \partial B_\beta \setminus \{1\}\}$ and the assumption (2.9) imply that this integral is

absolutely convergent and defines an element of $B(X)$. It does not depend on β and the mapping

$$H_0^\infty(B_\gamma) \longrightarrow B(X), \quad \varphi \mapsto \varphi(T),$$

is an algebra homomorphism. Proofs of these facts are similar to the ones in the sectorial case.

We state a technical observation for further use.

Lemma 2.3. *Let T be a Ritt operator of type α . Then rT is a Ritt operator for any $r \in (0, 1)$ and:*

(1) *For any $\beta \in (\alpha, \frac{\pi}{2})$, the set*

$$\{(\lambda - 1)R(\lambda, rT) : r \in (0, 1), \lambda \in \mathbb{C} \setminus B_\beta\}$$

is bounded;

(2) *For any $\gamma \in (\alpha, \frac{\pi}{2})$ and any $\varphi \in H_0^\infty(B_\gamma)$, $\varphi(T) = \lim_{r \rightarrow 1^-} \varphi(rT)$.*

Proof. Consider $\beta \in (\alpha, \frac{\pi}{2})$. It is clear that for any $\lambda \in \mathbb{C} \setminus B_\beta$ and any $r \in (0, 1)$, $\frac{\lambda}{r} \in \mathbb{C} \setminus \overline{B_\beta}$, $\lambda \notin \sigma(rT)$ and we have

$$(\lambda - 1)R(\lambda, rT) = \frac{\lambda - 1}{\lambda - r} \left(\frac{\lambda}{r} - 1 \right) R\left(\frac{\lambda}{r}, T\right)$$

Since the sets

$$\{(\lambda - 1)(\lambda - r)^{-1} : r \in (0, 1), \lambda \in \mathbb{C} \setminus B_\beta\}$$

and

$$\{(\mu - 1)R(\mu, T) : \mu \in \mathbb{C} \setminus \overline{B_\beta}\}$$

are bounded, we obtain (1).

Applying Lebesgue's Theorem to (2.10), the assertion (2) follows at once. \square

Let $H_{0,1}^\infty(B_\gamma) \subset H^\infty(B_\gamma)$ be the linear span of $H_0^\infty(B_\gamma)$ and constant functions. For any $\varphi = c + \psi$, with $c \in \mathbb{C}$ and $\psi \in H_0^\infty(B_\gamma)$, set $\varphi(T) = cI_X + \psi(T)$. Then $H_{0,1}^\infty(B_\gamma) \subset H^\infty(B_\gamma)$ is a unital algebra and $\varphi \mapsto \varphi(T)$ is a unital homomorphism from $H_{0,1}^\infty(B_\gamma)$ into $B(X)$. Note that $H_{0,1}^\infty(B_\gamma)$ contains rational functions with poles off $\overline{B_\gamma}$, and hence polynomials.

For any T as above and any $r \in (0, 1)$, $\sigma(rT) = r\sigma(T) \subset B_\beta$, hence the definition of $\varphi(rT)$ provided by (2.10) is given by the usual Dunford-Riesz functional calculus of rT . It therefore follows from classical properties of that functional calculus and the approximation Lemma 2.3 that for any rational function φ with poles off $\overline{B_\gamma}$, the above definition of $\varphi(T)$ coincides with the one obtained by substituting T to the complex variable. This applies in particular to any $\varphi \in \mathcal{P}$.

Likewise, recall that since $I - T$ is sectorial one can define its fractional powers $(I - T)^\delta$ for any $\delta > 0$. Then this bounded operator coincides with $\varphi_\delta(T)$, where φ_δ is the element of $H_0^\infty(B_\gamma)$ given by $\varphi_\delta(\lambda) = (1 - \lambda)^\delta$. See [39, Section 6] and [19, Chapter 3] for similar results.

Definition 2.4. *Let T be a Ritt operator of type α and let $\gamma \in (\alpha, \frac{\pi}{2})$. We say that T admits a bounded $H^\infty(B_\gamma)$ functional calculus if there exists a constant $K > 0$ such that*

$$\|\varphi(T)\| \leq K \|\varphi\|_{\infty, B_\gamma}, \quad \varphi \in H_0^\infty(B_\gamma).$$

In this case $\varphi \mapsto \varphi(T)$ is a bounded homomorphism on $H_{0,1}^\infty(B_\gamma)$. The next statement shows that the above functional calculus property can be tested on polynomials only.

Proposition 2.5. *A Ritt operator T has a bounded $H^\infty(B_\gamma)$ functional calculus if and only if there exists a constant $K \geq 1$ such that*

$$(2.11) \quad \|\varphi(T)\| \leq K \|\varphi\|_{\infty, B_\gamma}$$

for any $\varphi \in \mathcal{P}$.

Proof. The ‘only if’ part is clear from the above discussion. To prove the ‘if’ part, assume (2.11) on \mathcal{P} and consider $\varphi \in H_0^\infty(B_\gamma)$. Let $r \in (0, 1)$, let $r' \in (r, 1)$ be an auxiliary real number and let Γ be the boundary of $r'B_\gamma$ oriented counterclockwise.

By Runge’s Theorem (see e.g. [50, Thm 13.9]), there exists a sequence $(\varphi_m)_{m \geq 1}$ of polynomials such that $\varphi_m \rightarrow \varphi$ uniformly on the compact set $r'\overline{B_\gamma}$. Since $\sigma(rT) \subset r'B_\gamma$, we deduce that

$$\varphi_m(rT) = \frac{1}{2\pi i} \int_\Gamma \varphi_m(\lambda) R(\lambda, rT) d\lambda \longrightarrow \frac{1}{2\pi i} \int_\Gamma \varphi(\lambda) R(\lambda, rT) d\lambda = \varphi(rT),$$

when $m \rightarrow \infty$. By (2.11),

$$\|\varphi_m(rT)\| \leq K \|\varphi_m\|_{\infty, rB_\gamma} \leq K \|\varphi_m\|_{\infty, r'B_\gamma}.$$

Passing to the limit yields

$$\|\varphi(rT)\| \leq K \|\varphi\|_{\infty, r'B_\gamma}.$$

Finally letting $r \rightarrow 1$ and applying Lemma 2.3, (2), we deduce $\|\varphi(T)\| \leq K \|\varphi\|_{\infty, B_\gamma}$. \square

The above result is closely related to the following classical notion.

Definition 2.6. *We say that a bounded operator $T: X \rightarrow X$ is polynomially bounded if there is a constant $K \geq 1$ such that*

$$\|\varphi(T)\| \leq K \|\varphi\|_{\infty, \mathbb{D}}, \quad \varphi \in \mathcal{P}.$$

Obviously any Ritt operator with a bounded $H^\infty(B_\gamma)$ functional calculus is polynomially bounded. See Proposition 7.6 below for a partial converse.

According to [30, Prop. 5.2], there exist Ritt operators on Hilbert space which are not polynomially bounded. Thus there exist Ritt operators without any bounded $H^\infty(B_\gamma)$ functional calculus. Note that various such (counter-)examples can be derived from Proposition 4.1 below or from our Section 8.a.

Remark 2.7. Let T be a Ritt operator of type α , let $\gamma \in (\alpha, \frac{\pi}{2})$, and assume that $I - T$ has a dense range. Then $I - T$ is 1-1 by [10, Thm. 3.8] and arguing as in [10, 39], one can extend the definition of $\varphi(T)$ to any $\varphi \in H^\infty(B_\gamma)$. Namely let $\psi(z) = 1 - z$ and for any $\varphi \in H^\infty(B_\gamma)$, set $\varphi(T) = (I - T)^{-1}(\varphi\psi)(T)$, where $(\varphi\psi)(T)$ is defined by (2.10) and $\varphi(T)$ is defined on $D(\varphi(T)) = \{x \in X : (\varphi\psi)(T)x \in \text{Ran}(I - T)\}$. It is easy to check that the domain of $\varphi(T)$ contains $\text{Ran}(I - T)$, so that $\varphi(T)$ is densely defined, and that $\varphi(T)$ is closed. Consequently, $\varphi(T)$ is bounded if and only if $D(\varphi(T)) = X$.

Assume that T has a bounded $H^\infty(B_\gamma)$ functional calculus. Then $\varphi(T)$ is bounded for any $\varphi \in H^\infty(B_\gamma)$. Indeed let $\psi_n(z) = (1 - z)((1 - z) + n^{-1})^{-1}$ for any integer $n \geq 1$. The

sectoriality of $(I - T)$ ensures that $(\psi_n(T))_{n \geq 1}$ is bounded. Hence there is a constant $K > 0$ such that $\|(\varphi\psi_n)(T)\| \leq K\|\varphi\|_{\infty, B_\gamma}$ for any $n \geq 1$. It is easy to check that $(\varphi\psi_n)(T)x \rightarrow \varphi(T)x$ for any $x \in \text{Ran}(I - T)$. This shows the boundedness of $\varphi(T)$, with the estimate $\|\varphi(T)\| \leq K\|\varphi\|_{\infty, B_\gamma}$.

Remark 2.8. It is clear that the adjoint $T^*: X^* \rightarrow X^*$ of a Ritt operator $T \in B(X)$ (of type α) is a Ritt operator (of type α) as well. In this case, $\varphi(T)^* = \varphi(T^*)$ for any $\varphi \in H_0^\infty(B_\gamma)$ with $\gamma > \alpha$. Hence T^* has a bounded $H^\infty(B_\gamma)$ functional calculus if and only if T has one.

3. SQUARE FUNCTIONS

On general Banach spaces, square functions of the form (1.2) or (1.3) need to be replaced by suitable Rademacher averages. This short section is devoted to precise definitions of these abstract square functions, as well as to relevant properties of Rademacher norms on certain Banach spaces.

We let $(\varepsilon_k)_{k \geq 1}$ be a sequence of independent Rademacher variables on some probability space $(\mathcal{M}, d\mathbb{P})$. Given any Banach space X , we let $\text{Rad}(X)$ denote the closed subspace of the Bochner space $L^2(\mathcal{M}; X)$ spanned by the set $\{\varepsilon_k \otimes x : k \geq 1, x \in X\}$. Thus for any finite family $(x_k)_{k \geq 1}$ of elements of X ,

$$(3.1) \quad \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)} = \left(\int_{\mathcal{M}} \left\| \sum_k \varepsilon_k(u) x_k \right\|_X^2 d\mathbb{P}(u) \right)^{\frac{1}{2}}.$$

Moreover elements of $\text{Rad}(X)$ are sums of convergent series of the form $\sum_{k=1}^{\infty} \varepsilon_k \otimes x_k$.

For any bounded operator $T: X \rightarrow X$, for any integer $m \geq 1$ and for any $x \in X$, we set

$$\|x\|_{T,m} = \left\| \sum_{k=1}^{\infty} k^{m-\frac{1}{2}} \varepsilon_k \otimes T^{k-1}(I - T)^m(x) \right\|_{\text{Rad}(X)}.$$

More precisely for any $x \in X$ and any integer $k \geq 1$, set $x_k = k^{m-\frac{1}{2}} T^{k-1}(I - T)^m(x)$. Then $\|x\|_{T,m}$ is equal to the $\text{Rad}(X)$ -norm of $\sum_{k=1}^{\infty} \varepsilon_k \otimes x_k$ if this series converges in $L^2(\mathcal{M}; X)$, and $\|x\|_{T,m} = \infty$ otherwise.

If $X = L^p(\Omega)$ for some $1 \leq p < \infty$, then we have an equivalence

$$(3.2) \quad \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(L^p(\Omega))} \approx \left\| \left(\sum_k |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

for finite families $(x_k)_k$ of X (see e.g. [35, Thm. 1.d.6]). Hence for any $T: L^p(\Omega) \rightarrow L^p(\Omega)$ and any $m \geq 1$, we have

$$(3.3) \quad \|x\|_{T,m} \approx \left\| \left(\sum_{k=1}^{\infty} k^{2m-1} |T^{k-1}(I - T)^m(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}, \quad x \in L^p(\Omega).$$

In particular, the square function $\|\cdot\|_T$ defined by (1.3) is equivalent to $\|\cdot\|_{T,1}$.

Likewise, the Rademacher average (3.1) of a finite sequence $(x_k)_k$ on Hilbert space H is equal to $(\sum_k \|x_k\|_H^2)^{\frac{1}{2}}$, hence for any $T \in B(H)$, we have

$$\|x\|_{T,m} = \left(\sum_{k=1}^{\infty} k^{2m-1} \|T^{k-1}(I-T)^m(x)\|^2 \right)^{\frac{1}{2}}, \quad x \in H.$$

Square functions appearing in (3.3) are analogs of well-known square functions associated to sectorial operators on L^p -spaces. Namely let A be a sectorial operator of type $< \frac{\pi}{2}$ on $L^p(\Omega)$. Then $-A$ generates a bounded analytic semigroup $(e^{-tA})_{t \geq 0}$ on $L^p(\Omega)$ and for any integer $m \geq 1$, one may consider

$$\|x\|_{A,m} = \left\| \left(\int_0^\infty t^{2m-1} |A^m e^{-tA}(x)|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}, \quad x \in L^p(\Omega).$$

For any $t > 0$, $\frac{\partial^m}{\partial t^m}(e^{-tA}) = (-1)^m A^m e^{-tA}$. Hence if we regard $(T^{k-1}(I-T)^m)_{k \geq 1}$ as the m -th discrete derivative of the sequence $(T^{k-1})_{k \geq 1}$, then $\|\cdot\|_{T,m}$ is the discrete analog of the continuous square function $\|\cdot\|_{A,m}$. Thus Theorem 1.1 is a discrete analog of the main result of [10] showing the equivalence between the boundedness of H^∞ -functional calculus and square function estimates for sectorial operators.

Similar comments apply to the Hilbert space case.

In the sequel, the square functions $\|\cdot\|_{T,m}$ will be used for Ritt operators (although their definitions make sense for any operator).

Let X be a Banach space. The space $\text{Rad}(\text{Rad}(X))$ is the closure of finite sums

$$\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}$$

in $L^2(\mathcal{M} \times \mathcal{M}; X)$, where $x_{ij} \in X$ for any $i, j \geq 1$. We say that X has property (α) if the above decomposition is unconditional, that is, there exists a constant $C > 0$ such that for any finite family $(x_{ij})_{i,j \geq 1}$ of X and any family $(t_{ij})_{i,j \geq 1}$ of complex numbers,

$$\left\| \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes t_{ij} x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \leq C \sup_{i,j} |t_{ij}| \left\| \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))}.$$

Classical L^p -spaces (for $p < \infty$) have property (α) , indeed we have an equivalence

$$(3.4) \quad \left\| \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(L^p(\Omega)))} \approx \left\| \left(\sum_{i,j} |x_{ij}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

for finite families $(x_{ij})_{i,j}$ of $L^p(\Omega)$, which extends (3.2). This actually holds true as well for any Banach lattice with a finite cotype in place of $L^p(\Omega)$.

On the contrary, infinite dimensional noncommutative L^p -spaces (for $p \neq 2$) do not have property (α) . This goes back to [44], where property (α) was introduced.

We shall now supply more precise information, namely the so-called noncommutative Khintchine inequalities in one or two variables. In the one-variable case, these inequalities, stated as (3.5) and (3.6) below are due to Lust-Piquard for $1 < p < \infty$ [36] and Lust-Piquard

and Pisier for $p = 1$ [37]. The two-variable inequalities (3.7) and (3.8) are taken from [45, pp. 111-112].

In the sequel we let M be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace and for any $1 \leq p \leq \infty$, we let $L^p(M)$ denote the associated noncommutative L^p -space. We refer the reader to [48] for background and general information on these spaces. Any element of $L^p(M)$ is a (possibly unbounded) operator and for any such x , the modulus of x used in the next formulas will be

$$|x| = (x^*x)^{\frac{1}{2}}.$$

The following equivalences, valid for finite families of $L^p(M)$, are the noncommutative counterpart of (3.2). If $2 \leq p < \infty$, then

$$(3.5) \quad \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(L^p(M))} \approx \max \left\{ \left\| \left(\sum_k |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)}, \left\| \left(\sum_k |x_k^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \right\}.$$

If $1 \leq p \leq 2$, then

$$(3.6) \quad \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(L^p(M))} \approx \inf \left\{ \left\| \left(\sum_k |u_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} + \left\| \left(\sum_k |v_k^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \right\},$$

where the infimum runs over all possible decompositions $x_k = u_k + v_k$ in $L^p(M)$.

Let $n \geq 1$ be an integer. The space $L^p(M_n(M))$ associated to the von Neumann algebra $M_n(M)$ can be canonically identified with the vector space of all $n \times n$ matrices with entries in $L^p(M)$. The following equivalences are the noncommutative counterpart of (3.4). If $2 \leq p < \infty$, then

$$(3.7) \quad \left\| \sum_{i,j=1}^n \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(L^p(M)))} \approx \max \left\{ \left\| \left(\sum_{i,j=1}^n |x_{ij}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)}, \left\| \left(\sum_{i,j=1}^n |x_{ij}^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)}, \right. \\ \left. \left\| [x_{ij}] \right\|_{L^p(M_n(M))}, \left\| [x_{ji}] \right\|_{L^p(M_n(M))} \right\}.$$

If $1 \leq p \leq 2$, then

$$(3.8) \quad \left\| \sum_{i,j=1}^n \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(L^p(M)))} \approx \inf \left\{ \left\| \left(\sum_{i,j=1}^n |u_{ij}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} + \left\| \left(\sum_{i,j=1}^n |v_{ij}^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \right. \\ \left. + \left\| [w_{ij}] \right\|_{L^p(M_n(M))} + \left\| [z_{ji}] \right\|_{L^p(M_n(M))} \right\},$$

where the infimum runs over all possible decompositions $x_{ij} = u_{ij} + v_{ij} + w_{ij} + z_{ij}$ in $L^p(M)$.

4. A TRANSFER PRINCIPLE FROM SECTORIAL OPERATORS TO RITT OPERATORS

Let $T: X \rightarrow X$ be a Ritt operator on an arbitrary Banach space. We noticed in Section 2 that

$$A = I - T$$

is a sectorial operator of type $< \frac{\pi}{2}$. The following transfer result will be extremely important for applications. Indeed it allows to apply known results from the theory of H^∞ -calculus for sectorial operators to our context. This principle will be illustrated in Section 8. The proof is a variant of the one of [18, Thm. 8.3], adapted to our situation (see also [33, Prop. 3.2]).

Proposition 4.1. *The following are equivalent.*

- (i) *T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*
- (ii) *A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$.*

Proof. It will be convenient to set

$$\Delta_\gamma = 1 - B_\gamma.$$

for any $\gamma \in (0, \frac{\pi}{2})$. This is a subset of the cone Σ_γ .

Assume (i). To any $f \in H_0^\infty(\Sigma_\gamma)$, associate φ given by $\varphi(\lambda) = f(1 - \lambda)$. Then φ is defined on B_γ , its restriction to that set belongs to $H_0^\infty(B_\gamma)$, and $\|\varphi\|_{\infty, B_\gamma} = \|f\|_{\infty, \Delta_\gamma} \leq \|f\|_{\infty, \Sigma_\gamma}$. Comparing (2.3) and (2.10) and applying Cauchy's Theorem, we see that

$$f(A) = \varphi(T).$$

These observations imply that A has a bounded $H^\infty(\Sigma_\gamma)$ functional calculus.

Assume conversely that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some θ in $(0, \frac{\pi}{2})$. It follows from Lemma 2.1 that

$$\sigma(A) \subset \overline{\Delta_\alpha}$$

for some $\alpha \in (0, \frac{\pi}{2})$. Taking θ close enough to $\frac{\pi}{2}$, we may assume that $\alpha < \theta$.

We fix $\gamma \in (\theta, \frac{\pi}{2})$ and choose an arbitrary $\beta \in (\theta, \gamma)$. Let Γ_1 be the juxtaposition of the segments $[\cos(\beta)e^{i\beta}, 0]$ and $[0, \cos(\beta)e^{-i\beta}]$. Then let Γ_2 be the curve going from $\cos(\beta)e^{-i\beta}$ to $\cos(\beta)e^{i\beta}$ counterclockwise along the circle of center 1 and radius $\sin(\beta)$. Thus

$$(4.1) \quad \partial\Delta_\beta = \{\Gamma_1, \Gamma_2\},$$

the juxtaposition of Γ_1 and Γ_2 (see Figure 2 below).

Let $\varphi \in H_0^\infty(B_\gamma)$ and let $f: \Delta_\gamma \rightarrow \mathbb{C}$ be the holomorphic function defined by

$$(4.2) \quad f(z) = \varphi(1 - z), \quad z \in \Delta_\gamma.$$

Then again we have $\|f\|_{\infty, \Delta_\gamma} = \|\varphi\|_{\infty, B_\gamma}$, moreover there exist two positive constants $c, s > 0$ such that

$$(4.3) \quad |f(z)| \leq c|z|^s, \quad z \in \Delta_\gamma.$$

We may define $f_1: \mathbb{C} \setminus \Gamma_1 \rightarrow \mathbb{C}$ and $f_2: \mathbb{C} \setminus \Gamma_2 \rightarrow \mathbb{C}$ by letting

$$(4.4) \quad f_1(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\lambda)}{\lambda - z} d\lambda \quad \text{and} \quad f_2(z) = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\lambda)}{\lambda - z} d\lambda.$$

Clearly these functions are holomorphic on their domains. According to (4.1) and Cauchy's Theorem, we have

$$(4.5) \quad \forall z \in \Delta_\beta, \quad f(z) = f_1(z) + f_2(z).$$

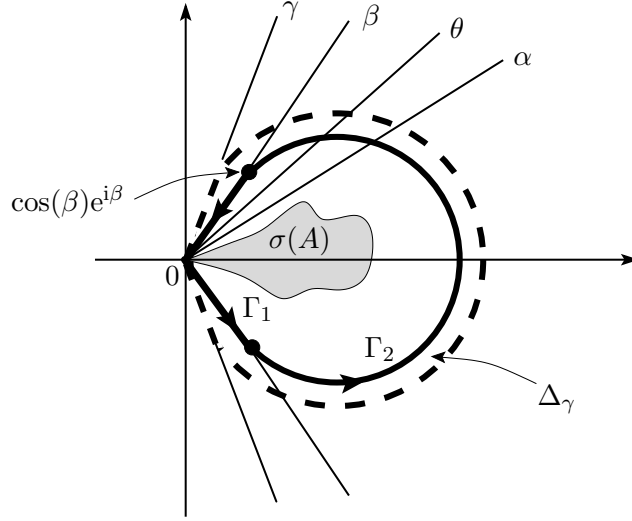


FIGURE 2.

Since the distance between Γ_1 and $\Sigma_\theta \setminus \Delta_\theta$ is strictly positive and $\Gamma_1 \subset \Delta_\gamma$, there is a constant $C_1 \geq 0$ (not depending on f) such that

$$(4.6) \quad \forall z \in \Sigma_\theta \setminus \Delta_\theta, \quad |f_1(z)| \leq C_1 \|f\|_{\infty, \Delta_\gamma}.$$

Likewise there is a constant $C_2 \geq 0$ (not depending on f) such that

$$\forall z \in \Delta_\theta, \quad |f_2(z)| \leq C_2 \|f\|_{\infty, \Delta_\gamma}.$$

Combining with (4.5), this yields

$$\forall z \in \Delta_\theta, \quad |f_1(z)| \leq (1 + C_2) \|f\|_{\infty, \Delta_\gamma}.$$

Together with (4.6) this shows that $f_1 \in H^\infty(\Sigma_\theta)$ and that with $C_3 = \max\{C_1, 1 + C_2\}$, we have

$$(4.7) \quad \|f_1\|_{\infty, \Sigma_\theta} \leq C_3 \|f\|_{\infty, \Delta_\gamma}.$$

Now let $g: \Sigma_\theta \rightarrow \mathbb{C}$ be defined by

$$g(z) = f_1(z) + \frac{f_2(0)}{1+z}.$$

According to the definition of f_1 given by (4.4), $zf_1(z)$ is bounded when $|z| \rightarrow \infty$. Hence $zg(z)$ is bounded on Σ_θ . Further, f_2 is defined about 0, hence $|f_2(z) - f_2(0)| \lesssim |z|$ on Δ_θ . By (4.5), we have

$$g(z) = f(z) + \left(\frac{f_2(0)}{1+z} - f_2(z) \right) = f(z) + (f_2(0) - f_2(z)) - f_2(0) \frac{z}{1+z}$$

on Δ_θ . Applying the above estimate and (4.3), we deduce that $|g(z)| \lesssim \max\{|z|^s, |z|\}$ on Δ_θ . These estimates show that g belongs to $H_0^\infty(\Sigma_\theta)$. We may therefore compute $g(A)$ by means of (2.3), and hence $f_1(A)$ by

$$f_1(A) = g(A) - f_2(0)(I + A)^{-1}.$$

From the assumption (ii), we get a constant $C_4 \geq 0$ (not depending on f) such that

$$\|f_1(A)\| \leq C_4 \|f_1\|_{\infty, \Sigma_\theta}.$$

Combining with (4.7), we deduce

$$\|f_1(A)\| \leq C_3 C_4 \|f\|_{\infty, \Delta_\gamma}.$$

The holomorphic function f_2 is defined on an open neighborhood of the spectrum $\sigma(A)$. Hence $f_2(A)$ may be defined by the classical Riesz-Dunford functional calculus. Then by Fubini's Theorem and (4.4), we have

$$f_2(A) = \frac{1}{2\pi i} \int_{\Gamma_2} f(\lambda) R(\lambda, A) d\lambda.$$

Consequently,

$$\|f_2(A)\| \leq \frac{1}{2\pi} \int_{\Gamma_2} |f(\lambda)| \|R(\lambda, A)\| |d\lambda|.$$

We deduce that there is a constant $C_5 \geq 0$ (not depending on f) such that

$$\|f_2(A)\| \leq C_5 \|f\|_{\infty, \Delta_\gamma}.$$

Using (4.2) and (4.5), it is easy to check that

$$\varphi(T) = f_1(A) + f_2(A).$$

We deduce (with $C = C_3 C_4 + C_5$) an estimate

$$\|\varphi(T)\| \leq C \|\varphi\|_{\infty, B_\gamma},$$

which shows the boundedness of the $H^\infty(B_\gamma)$ functional calculus. \square

Remark 4.2. We mention another (easier) transfer principle. Let $(T_t)_{t \geq 0}$ be a bounded analytic semigroup on X , and let $-A$ denote its infinitesimal generator. For any fixed $t \geq 0$, T_t is a Ritt operator; this is easy to check, see [53, Section 3] for more on this. Writing $\varphi(T_t) = f(A)$ with $f(z) = \varphi(e^{-tz})$, one shows that if A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$, then T_t admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.

5. R -BOUNDEDNESS AND R -RITT OPERATORS

This section starts with some background on R -boundedness, a notion which -by now- plays a prominent role in many questions concerning functional calculi, see in particular [24, 25, 54]. R -boundedness was introduced in [6] and significantly developed in [9]. The resulting notion of R -Ritt operator (see below) was first studied by Blunck [7, 8].

Let X be a Banach space and let $E \subset B(X)$ be a set of bounded operators on X . We say that E is R -bounded provided that there exists a constant $C \geq 0$ such that for any finite family $(T_k)_k$ of E and any finite family $(x_k)_k$ of X ,

$$\left\| \sum_k \varepsilon_k \otimes T_k(x_k) \right\|_{\text{Rad}(X)} \leq C \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)}.$$

In this case, we let $\mathcal{R}(E)$ denote the smallest possible C . Any R -bounded set E is bounded, with $\|T\| \leq \mathcal{R}(E)$ for any $T \in E$. If $X = H$ is a Hilbert space, the converse holds true, because of the isometric isomorphism $\text{Rad}(H) = \ell^2(H)$. But if X is not isomorphic to a Hilbert space, then the unit ball of $B(X)$ is not R -bounded (see [1]).

We will use the following convexity result. This is a well-known consequence of [9, Lem. 3.2], see also [21, Lem. 4.2].

Lemma 5.1. *Let $J \subset \mathbb{R}$ be an interval, let $E \subset B(X)$ be an R -bounded set and let $K > 0$ be a constant. Then the set*

$$E_K = \left\{ \int_J h(t)F(t) dt \mid F: J \rightarrow E \text{ is continuous, } h \in L^1(J; dt) \text{ and } \int_J |h(t)| dt \leq K \right\}$$

is R -bounded, with $\mathcal{R}(E_K) \leq 2K\mathcal{R}(E)$.

A sectorial operator A on X is called R -sectorial of R -type ω provided that $\sigma(A) \subset \overline{\Sigma_\omega}$ and for any $\nu \in (\omega, \pi)$, the set (2.2) is R -bounded.

Likewise, a Ritt operator T on X is called R -Ritt provided that the two sets in (2.4) are R -bounded. The following is an ‘ R -bounded’ version of (2.5) and Lemma 2.1. We refer to [7] for closely related results.

Lemma 5.2. *Let $T: X \rightarrow X$ be a Ritt operator and let $A = I - T$. The following are equivalent.*

- (i) T is R -Ritt.
- (ii) A is R -sectorial of R -type $< \frac{\pi}{2}$.
- (iii) *There exists an angle $\alpha \in (0, \frac{\pi}{2})$ such that $\sigma(T) \subset \overline{B_\alpha}$ and for any $\beta \in (\alpha, \frac{\pi}{2})$, the set*

$$(5.1) \quad \{(\lambda - 1)R(\lambda, T) : \lambda \in \mathbb{C} \setminus \overline{B_\beta}\}$$

is R -bounded.

Proof. The implications ‘(i) \Rightarrow (ii)’ and ‘(iii) \Rightarrow (i)’ follow from [7]. The proof of ‘(ii) \Rightarrow (iii)’ is parallel to the one of Lemma 2.1, using two elementary but important results on R -boundedness due to L. Weis. The first one says that for any open set $\emptyset \subset \mathbb{C}$ and for any compact set $F \subset \emptyset$, any analytic function $\emptyset \rightarrow B(X)$ maps F into an R -bounded subset of $B(X)$ [54, Prop. 2.6]. With the notation of the proof of Lemma 2.1, this implies that the two sets

$$E_1 = h(\Lambda_\beta) \quad \text{and} \quad E_2 = \{h(\lambda) : |\lambda| = 2\}$$

are R -bounded. The second one is the ‘maximum principle’ for R -boundedness [54, Prop. 2.8]. Together with the R -boundedness of E_2 , it implies that $\{h(\lambda) : |\lambda| \geq 2\}$ is R -bounded. With these elements in hand, the adaptation of the proof of Lemma 2.1 is straightforward. \square

We will say that T is an R -Ritt operator of R -type α if it satisfies condition (iii) of Lemma 5.2. It is clear that in this case, $A = I - T$ is R -sectorial of R -type α .

In the rest of this section, we are going to focus on *commutative* L^p -spaces, see however Remark 5.5. Our objective is the following theorem, which is a key step in our proof of Theorem 1.1.

Theorem 5.3. *Let (Ω, μ) be a measure space, let $1 < p < \infty$ and let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a power bounded operator. Assume that it satisfies uniform estimates*

$$(5.2) \quad \|x\|_{T,1} \lesssim \|x\|_{L^p} \quad \text{and} \quad \|y\|_{T^*,1} \lesssim \|y\|_{L^{p'}}$$

for $x \in L^p(\Omega)$ and $y \in L^{p'}(\Omega)$. Then T is R-Ritt.

Until the end of the proof of this theorem, we fix a bounded operator $T: L^p(\Omega) \rightarrow L^p(\Omega)$, with $1 < p < \infty$. The following lemma is inspired by the proof of [23, Thm. 4.7].

Lemma 5.4. *If T satisfies a uniform estimate*

$$(5.3) \quad \|x\|_{T,1} \lesssim \|x\|, \quad x \in L^p(\Omega),$$

then it automatically satisfies a uniform estimate

$$(5.4) \quad \|x\|_{T,2} \lesssim \|x\|, \quad x \in L^p(\Omega).$$

Proof. We will use the following elementary identity that the reader can easily check. For any integer $k \geq 1$,

$$(5.5) \quad \sum_{j=1}^k j(k+1-j) = \frac{1}{6} k(k+1)(k+2).$$

Let $x \in L^p(\Omega)$ and let $N \geq 1$ be an integer. According to the above identity we have a function inequality

$$\sum_{k=1}^N k^3 |T^{k-1}(I-T)^2 x|^2 \leq 6 \sum_{k=1}^N \sum_{j=1}^k j(k+1-j) |T^{k-1}(I-T)^2 x|^2.$$

By a change of indices (letting $r = k+1-j$ for any fixed j), we have

$$\begin{aligned} \sum_{k=1}^N \sum_{j=1}^k j(k+1-j) |T^{k-1}(I-T)^2 x|^2 &= \sum_{j=1}^N j \sum_{k=j}^N (k+1-j) |T^{k-1}(I-T)^2 x|^2 \\ &= \sum_{j=1}^N j \sum_{r=1}^{N+1-j} r |T^{r+j-2}(I-T)^2 x|^2 \\ &\leq \sum_{j=1}^N j \sum_{r=1}^N r |T^{r+j-2}(I-T)^2 x|^2. \end{aligned}$$

According to (3.4), we have an estimate

$$\left\| \left(\sum_{j,r=1}^N jr |T^{r+j-2}(I-T)^2 x|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \lesssim \left\| \left(\sum_{j,r=1}^N j^{\frac{1}{2}} r^{\frac{1}{2}} \varepsilon_j \otimes \varepsilon_r \otimes T^{r+j-2}(I-T)^2 x \right) \right\|_{\text{Rad}(\text{Rad}(L^p(\Omega)))}.$$

Furthermore, writing

$$T^{r+j-2}(I-T)^2 x = T^{j-1}(I-T)[T^{r-1}(I-T)x],$$

and applying the assumption (5.3) twice, we see that

$$\begin{aligned} \left\| \sum_{j,r=1}^N j^{\frac{1}{2}} r^{\frac{1}{2}} \varepsilon_j \otimes \varepsilon_r \otimes T^{r+j-2} (I - T)^2 x \right\|_{\text{Rad}(\text{Rad}(L^p(\Omega)))} &\lesssim \left\| \sum_{r=1}^N r^{\frac{1}{2}} \varepsilon_r \otimes T^{r-1} (I - T) x \right\|_{\text{Rad}(L^p(\Omega))} \\ &\lesssim \|x\|. \end{aligned}$$

Altogether, we obtain the estimate

$$\left\| \left(\sum_{k=1}^N k^3 |T^{k-1} (I - T)^2 x|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \lesssim \|x\|,$$

which proves (5.4). \square

Proof of Theorem 5.3. Since T is power bounded and $X = L^p(\Omega)$ is reflexive, the Mean Ergodic Theorem ensures that

$$(5.6) \quad X = \text{Ker}(I - T) \oplus \overline{\text{Ran}(I - T)}.$$

Furthermore the two square function estimates (5.2) imply that

$$\|x\| \approx \|x\|_{T,1}, \quad x \in \overline{\text{Ran}(I - T)}.$$

Indeed this is implicit in [33, Cor. 3.4], to which we refer for details. Let $(x_n)_{n \geq 1}$ be a finite family of $\overline{\text{Ran}(I - T)}$, and let $(\eta_n)_{n \geq 1}$ be a sequence of ± 1 . The above equivalence yields

$$\left\| \sum_{n \geq 1} \eta_n x_n \right\|_{L^p(\Omega)} \approx \left\| \sum_{k \geq 1} \sum_{n \geq 1} k^{\frac{1}{2}} \eta_n \varepsilon_k \otimes T^{k-1} (I - T) x_n \right\|_{\text{Rad}(L^p(\Omega))}.$$

Averaging over the $\eta_n = \pm 1$ and applying (3.4), we obtain that

$$(5.7) \quad \left\| \sum_{n \geq 1} \varepsilon_n \otimes x_n \right\|_{\text{Rad}(L^p(\Omega))} \approx \left\| \left(\sum_{k,n \geq 1} k |T^{k-1} (I - T) x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

for x_n in $\overline{\text{Ran}(I - T)}$.

Applying Lemma 5.4 and similarly averaging the resulting estimates

$$\left\| \sum_n \eta_n x_n \right\|_{T,2} \lesssim \left\| \sum_n \eta_n x_n \right\|$$

over all $\eta_n = \pm 1$, we obtain that

$$(5.8) \quad \left\| \left(\sum_{k,n \geq 1} k^3 |T^{k-1} (I - T)^2 x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \lesssim \left\| \sum_{n \geq 1} \varepsilon_n \otimes x_n \right\|_{\text{Rad}(L^p(\Omega))}$$

for x_n in $L^p(\Omega)$.

Our aim is to show that the two sets in (2.4) are R -bounded. Their restrictions to the kernel $\text{Ker}(I - T)$ clearly have this property. By (5.6) it therefore suffices to consider their restrictions to $\overline{\text{Ran}(I - T)}$.

Let $(x_n)_{n \geq 1}$ be a finite family of $\overline{\text{Ran}(I - T)}$. Each $T^n x_n$ belongs to that space hence by (5.7), we have

$$\left\| \sum_{n \geq 1} \varepsilon_n \otimes T^n x_n \right\|_{\text{Rad}(L^p(\Omega))} \lesssim \left\| \left(\sum_{k, n \geq 1} k |T^{k+n-1}(I - T)x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}.$$

Moreover

$$\begin{aligned} \left\| \left(\sum_{k, n \geq 1} k |T^{k+n-1}(I - T)x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} &\leq \left\| \left(\sum_{k, n \geq 1} (k + n) |T^{k+n-1}(I - T)x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\ &\leq \left\| \left(\sum_{n \geq 1} \sum_{k \geq n+1} k |T^{k-1}(I - T)x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\ &\leq \left\| \left(\sum_{k, n \geq 1} k |T^{k-1}(I - T)x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}. \end{aligned}$$

Using (5.7) we deduce that

$$\left\| \sum_{n \geq 1} \varepsilon_n \otimes T^n x_n \right\|_{\text{Rad}(L^p(\Omega))} \lesssim \left\| \sum_{n \geq 1} \varepsilon_n \otimes x_n \right\|_{\text{Rad}(L^p(\Omega))}.$$

This shows the R -boundedness of $\{T^n : n \geq 1\}$.

Likewise using (5.7) and (5.8), we have

$$\begin{aligned} \left\| \sum_{n \geq 1} \varepsilon_n \otimes n T^{n-1}(I - T)x_n \right\|_{\text{Rad}(L^p(\Omega))} &\lesssim \left\| \left(\sum_{k, n \geq 1} k |n T^{k+n-2}(I - T)^2 x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\ &\lesssim \left\| \left(\sum_{k, n \geq 1} (k + n)^3 |T^{k+n-2}(I - T)^2 x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\ &\lesssim \left\| \left(\sum_{n \geq 1} \sum_{k \geq n} (k + 1)^3 |T^{k-1}(I - T)^2 x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\ &\lesssim \left\| \left(\sum_{k, n \geq 1} k^3 |T^{k-1}(I - T)^2 x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \\ &\lesssim \left\| \sum_{n \geq 1} \varepsilon_n \otimes x_n \right\|_{\text{Rad}(L^p(\Omega))}. \end{aligned}$$

Thus the set $\{n T^{n-1}(I - T) : n \geq 1\}$ is R -bounded as well, which completes the proof. \square

Remark 5.5. It is easy to check that the above proof and hence Theorem 5.3 extend to the case when $L^p(\Omega)$ is replaced by a reflexive Banach space with property (α) . In particular this holds true on any reflexive Banach lattice with finite cotype. However we do not know whether Theorem 5.3 holds true on *noncommutative* L^p -spaces.

The above proof can be adapted to the sectorial case, which yields a slight improvement of the main result of [10]. We will explain this point in a separate note [32].

6. FROM H^∞ FUNCTIONAL CALCULUS TO SQUARE FUNCTIONS

The main aim of this section is to decide whether a Ritt operator T with a bounded $H_0^\infty(B_\gamma)$ functional calculus necessarily satisfies square function estimates $\|x\|_{T,m} \lesssim \|x\|$. We will show that this holds true on a large class of Banach spaces, including spaces with property (α) and noncommutative L^p -spaces for $p < \infty$.

For that purpose, we investigate a strong form of bounded holomorphic functional calculus which is somehow natural in order to make connections with square functions. We consider both the sectorial case and the Ritt case.

Let f_1, \dots, f_n be a finite family of $H^\infty(\emptyset)$, for some non empty open set $\emptyset \subset \mathbb{C}$. In the sequel we let

$$\left\| \left(\sum_{l=1}^n |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \emptyset} = \sup \left\{ \left(\sum_{l=1}^n |f_l(z)|^2 \right)^{\frac{1}{2}} : z \in \emptyset \right\}.$$

Equivalently, let (e_1, \dots, e_n) be the canonical basis of the Hermitian space ℓ_n^2 , then

$$(6.1) \quad \left\| \left(\sum_{l=1}^n |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \emptyset} = \left\| \sum_{l=1}^n f_l \otimes e_l \right\|_{H^\infty(\emptyset; \ell_n^2)}.$$

In the following definitions, X is an arbitrary Banach space.

Definition 6.1.

- (1) Let A be a sectorial operator of type $\omega \in (0, \pi)$ on X , and let $\theta \in (\omega, \pi)$. We say that A admits a quadratic $H^\infty(\Sigma_\theta)$ functional calculus if there exists a constant $K > 0$ such that for any $n \geq 1$, for any f_1, \dots, f_n in $H_0^\infty(\Sigma_\theta)$, and for any $x \in X$,

$$(6.2) \quad \left\| \sum_{l=1}^n \varepsilon_l \otimes f_l(A)x \right\|_{\text{Rad}(X)} \leq K \|x\| \left\| \left(\sum_{l=1}^n |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_\theta}.$$

- (2) Let T be a Ritt operator of type $\alpha \in (0, \frac{\pi}{2})$ on X , and let $\gamma \in (\alpha, \frac{\pi}{2})$. We say that T admits a quadratic $H^\infty(B_\gamma)$ functional calculus if there exists a constant $K > 0$ such that for any $n \geq 1$, for any $\varphi_1, \dots, \varphi_n$ in $H_0^\infty(B_\gamma)$, and for any $x \in X$,

$$\left\| \sum_{l=1}^n \varepsilon_l \otimes \varphi_l(T)x \right\|_{\text{Rad}(X)} \leq K \|x\| \left\| \left(\sum_{l=1}^n |\varphi_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, B_\gamma}.$$

Arguing as in Proposition 2.5, one can restrict to polynomials in Part (2).

It is clear that any sectorial operator with a quadratic $H^\infty(\Sigma_\theta)$ functional calculus has a bounded $H^\infty(\Sigma_\theta)$ functional calculus. We will see in Proposition 6.10 that the converse does not hold true. We are going to show however that up to a change of angle, that converse holds true on a large class of Banach spaces. For that purpose we introduce the following.

Definition 6.2. We say that a Banach space X has property (Q) if there exists a constant $C > 0$ such that

$$(6.3) \quad \left\| \sum_{k,l \geq 1} \alpha_{kl} \varepsilon_k \otimes \varepsilon_l \otimes x_k \right\|_{\text{Rad}(\text{Rad}(X))} \leq C \sup_k \left(\sum_l |\alpha_{kl}|^2 \right)^{\frac{1}{2}} \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)}$$

for any finite family $(\alpha_{kl})_{k,l \geq 1}$ of complex numbers and any finite family $(x_k)_{k \geq 1}$ of X .

Theorem 6.3. *Assume that X has property (Q) and let A be a sectorial operator on X with a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Then A admits a quadratic $H^\infty(\Sigma_\nu)$ functional calculus for any $\nu \in (\theta, \pi)$.*

Proof. The proof relies on a decomposition principle for holomorphic functions, due to E. Franks and A. McIntosh. Let $0 < \theta < \nu < \pi$ be two angles. That decomposition principle says that there exists a constant $C > 0$, and two sequences $(F_k)_{k \geq 1}$ and $(G_k)_{k \geq 1}$ in $H_0^\infty(\Sigma_\theta)$ such that:

- (a) For any $z \in \Sigma_\theta$, we have $\sum_{k \geq 1} |F_k(z)| \leq C$.
- (b) For any $z \in \Sigma_\theta$, we have $\sum_{k \geq 1} |G_k(z)| \leq C$.
- (c) For any Banach space Z and for any function $F \in H^\infty(\Sigma_\nu; Z)$, there exists a bounded sequence $(b_k)_{k \geq 1}$ in Z such that

$$\|b_k\| \leq C \|F\|_{H^\infty(\Sigma_\nu; Z)}, \quad k \geq 1,$$

and

$$F(z) = \sum_{k=1}^{\infty} b_k F_k(z) G_k(z), \quad z \in \Sigma_\theta.$$

Indeed, [15, Prop. 3.1] and the last paragraph of [15, Section 3] show this property for $Z = \mathbb{C}$. However it is easy to check that the proof works as well for Z -valued holomorphic functions.

Since A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, we have a uniform estimate

$$\left\| \sum_k \eta_k F_k(A) \right\| \lesssim \sup_{z \in \Sigma_\theta} \left| \sum_k \eta_k F_k(z) \right| \leq \sup_k |\eta_k| \sup_{z \in \Sigma_\theta} \sum_k |F_k(z)|$$

for finite families $(\eta_k)_{k \geq 1}$ of complex numbers. Hence by (a), we have

$$(6.4) \quad \sup_{m \geq 1} \sup_{\eta_k = \pm 1} \left\| \sum_{k=1}^m \eta_k F_k(A) \right\| < \infty.$$

Likewise, (b) implies that

$$(6.5) \quad \sup_{m \geq 1} \sup_{\eta_k = \pm 1} \left\| \sum_{k=1}^m \eta_k G_k(A) \right\| < \infty.$$

We will apply property (c) with $Z = \ell_n^2$ for arbitrary $n \geq 1$. Let f_1, \dots, f_n in $H_0^\infty(\Sigma_\nu)$, and consider

$$F = \sum_{l=1}^n f_l \otimes e_l \in H^\infty(\Sigma_\nu; \ell_n^2).$$

Let $(b_k)_{k \geq 1}$ be the bounded sequence of ℓ_n^2 provided by (c), and write $b_k = (\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kn})$ for any $k \geq 1$. Then

$$(6.6) \quad f_l(z) = \sum_{k=1}^{\infty} \alpha_{kl} F_k(z) G_k(z), \quad z \in \Sigma_\theta,$$

for any $l = 1, \dots, n$, and

$$(6.7) \quad \sup_k \left(\sum_l |\alpha_{kl}|^2 \right)^{\frac{1}{2}} \leq C \left\| \left(\sum_{l=1}^n |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_\nu}$$

by (c) and (6.1).

For any $l = 1, \dots, n$ and any integer $m \geq 1$, we consider the function

$$h_{m,l} = \sum_{k=1}^m \alpha_{kl} F_k G_k,$$

which belongs to $H_0^\infty(\Sigma_\nu)$ and approximates f_l by (6.6).

Let $x \in X$. By the Khintchine-Kahane inequality (see e.g. [35, Thm. 1.e.13]), we have

$$\begin{aligned} \left\| \sum_l \varepsilon_l \otimes h_{m,l}(A)x \right\|_{\text{Rad}(X)} &= \left(\int_{\mathcal{M}} \left\| \sum_{k,l} \varepsilon_l(u) \alpha_{kl} F_k(A) G_k(A)x \right\|^2 d\mathbb{P}(u) \right)^{\frac{1}{2}} \\ &\leq \int_{\mathcal{M}} \left\| \sum_k F_k(A) \left(\sum_l \varepsilon_l(u) \alpha_{kl} G_k(A)x \right) \right\| d\mathbb{P}(u). \end{aligned}$$

For any x_1, \dots, x_m in X , we have

$$\sum_k F_k(A)x_k = \int_{\mathcal{M}} \left(\sum_k \varepsilon_k(v) F_k(A) \right) \left(\sum_k \varepsilon_k(v) x_k \right) d\mathbb{P}(v),$$

hence

$$\begin{aligned} \left\| \sum_k F_k(A)x_k \right\| &\leq \int_{\mathcal{M}} \left\| \sum_k \varepsilon_k(v) F_k(A) \right\| \left\| \sum_k \varepsilon_k(v) x_k \right\| d\mathbb{P}(v) \\ &\lesssim \int_{\mathcal{M}} \left\| \sum_k \varepsilon_k(v) x_k \right\| d\mathbb{P}(v) \end{aligned}$$

by (6.4). Applying this estimate with $x_k = \sum_l \varepsilon_l(u) \alpha_{kl} G_k(A)x$ and integrating over $(u, v) \in \mathcal{M} \times \mathcal{M}$, we deduce that

$$\left\| \sum_l \varepsilon_l \otimes h_{m,l}(A)x \right\|_{\text{Rad}(X)} \lesssim \left\| \sum_{k,l} \alpha_{kl} \varepsilon_k \otimes \varepsilon_l \otimes G_k(A)x \right\|_{\text{Rad}(\text{Rad}(X))}.$$

By assumption, X has property (Q). Hence it follows from (6.7) and the above estimate that

$$\left\| \sum_l \varepsilon_l \otimes h_{m,l}(A)x \right\|_{\text{Rad}(X)} \lesssim \left\| \left(\sum_l |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_\nu} \left\| \sum_l \varepsilon_k \otimes G_k(A)x \right\|_{\text{Rad}(X)}.$$

Moreover according to (6.5), we have $\left\| \sum_k \varepsilon_k \otimes G_k(A)x \right\|_{\text{Rad}(X)} \lesssim \|x\|$. Thus we finally obtain

$$\left\| \sum_l \varepsilon_l \otimes h_{m,l}(A)x \right\|_{\text{Rad}(X)} \lesssim \|x\| \left\| \left(\sum_l |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_\nu}.$$

We deduce the expected result by an entirely classical approximation process, that we explain for the convenience of the reader. For any $\varepsilon \in (0, 1)$, set $A_\varepsilon = (\varepsilon I + A)(I + \varepsilon A)^{-1}$. Then A_ε is bounded and invertible, its spectrum is a compact subset of Σ_θ , and it follows from

Cauchy's Theorem that for some contour Γ_ε of finite length included in the open set Σ_θ , we have

$$h(A_\varepsilon) = \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} h(z) R(z, A_\varepsilon) dz$$

for any $h \in H_0^\infty(\Sigma_\theta)$. Since $h_{m,l} \rightarrow f_l$ pointwise and $\sup_{m,z} |h_{m,l}(z)| < \infty$, the above integral representation ensures that

$$\lim_{m \rightarrow \infty} h_{m,l}(A_\varepsilon) = f_l(A_\varepsilon), \quad l = 1, \dots, n.$$

Furthermore,

$$\lim_{\varepsilon \rightarrow 0} f_l(A_\varepsilon) = f_l(A)$$

for any $l = 1, \dots, n$, by [29, Lem. 2.4].

Now observe that the A_ε 's uniformly admit a bounded $H^\infty(\Sigma_\theta)$ functional calculus, that is, there exists a constant $K > 0$ such that $\|h(A_\varepsilon)\| \leq K \|h\|_{\infty, \Sigma_\theta}$ for any $h \in H_0^\infty(\Sigma_\theta)$ and any $\varepsilon \in (0, 1)$. It therefore follows from the above proof that there is a constant $K' > 0$ such that

$$(6.8) \quad \left\| \sum_l \varepsilon_l \otimes h_{m,l}(A_\varepsilon) x \right\|_{\text{Rad}(X)} \leq K' \|x\| \left\| \left(\sum_l |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_\nu}$$

for any $m \geq 1$ and any $\varepsilon \in (0, 1)$. Then (6.2) follows from (6.8). \square

We now state a similar result for Ritt operators and their functional calculus.

Theorem 6.4. *Assume that X has property (Q) and let T be a Ritt operator on X with a bounded $H^\infty(B_\gamma)$ functional calculus. Then T admits a quadratic $H^\infty(B_\nu)$ functional calculus for any $\nu \in (\gamma, \frac{\pi}{2})$.*

Proof. There are two ways to get to this result. The first one is to mimic the proof of Theorem 6.3, using a Franks-McIntosh decomposition adapted to Słolz domains. The existence of such decompositions follows from [15, Section 5].

The second way is to observe that the transfer principle stated as Proposition 4.1 holds true (with essentially the same proof) for the quadratic functional calculus. Namely with $A = I - T$, the following are equivalent:

- (i) T admits a quadratic $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.
- (ii) A admits a quadratic $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$.

Hence the result follows from Theorem 6.3, the implication '(i) \Rightarrow (ii)' of Proposition 4.1 and the implication '(ii) \Rightarrow (i)' above. \square

We shall now exhibit two classes of Banach spaces with property (Q), namely those with property (α) and the noncommutative L^p -spaces, for $p < \infty$.

According to [26, Lem. 4.3] and its proof, a Banach space X with property (α) satisfies the following property: there exists a constant $C > 0$ such that for any $n \geq 1$, for any finite

family $(b_k)_{k \geq 1}$ of elements of M_n that we denote by $b_k = [b_k(l, j)]_{1 \leq l, j \leq n}$ and for any n -tuple $(x_{k1})_{k \geq 1}, \dots, (x_{kn})_{k \geq 1}$ of families in X ,

$$(6.9) \quad \left\| \sum_{k \geq 1} \sum_{l, j=1}^n \varepsilon_k \otimes \varepsilon_l \otimes b_k(l, j) x_{kj} \right\|_{\text{Rad}(\text{Rad}(X))} \leq C \sup_k \|b_k\|_{M_n} \left\| \sum_{k, j} \varepsilon_k \otimes \varepsilon_j \otimes x_{kj} \right\|_{\text{Rad}(\text{Rad}(X))}.$$

If each b_k is a column matrix, i.e. $b_k(l, j) = 0$ for any $j = 2, \dots, n$, then

$$\|b_k\|_{M_n} = \left(\sum_{l=1}^n |b_k(l, 1)|^2 \right)^{\frac{1}{2}}$$

and the left handside of (6.9) is equal to

$$\left\| \sum_{k \geq 1} \sum_{l=1}^n b_k(l, 1) \varepsilon_k \otimes \varepsilon_l \otimes x_{k1} \right\|_{\text{Rad}(\text{Rad}(X))}.$$

We immediately deduce the following.

Proposition 6.5. *Banach spaces with property (α) have property (Q) .*

As a consequence of that proposition, Banach spaces with property (α) satisfy Theorems 6.3 and 6.4. However using the full strength of (6.9), we can obtain the following stronger results of independent interest.

Proposition 6.6. *Assume that X has property (α) .*

- (1) *Let A be a sectorial operator on X with a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Then for any $\nu \in (\theta, \pi)$, there exists a constant $K > 0$ such that*

$$(6.10) \quad \left\| \sum_{l, j=1}^n \varepsilon_l \otimes f_{lj}(A) x_j \right\|_{\text{Rad}(X)} \leq K \sup_{z \in \Sigma_\nu} \| [f_{lj}(z)] \|_{M_n} \left\| \sum_{j=1}^n \varepsilon_j \otimes x_j \right\|_{\text{Rad}(X)}$$

for any $n \geq 1$, for any matrix $[f_{lj}]$ of elements of $H_0^\infty(\Sigma_\nu)$ and for any x_1, \dots, x_n in X .

- (2) *Let T be a Ritt operator on X with a bounded $H^\infty(B_\gamma)$ functional calculus. Then for any $\nu \in (\gamma, \frac{\pi}{2})$, there exists a constant $K > 0$ such that*

$$(6.11) \quad \left\| \sum_{l, j=1}^n \varepsilon_l \otimes \varphi_{lj}(T) x_j \right\|_{\text{Rad}(X)} \leq K \sup_{z \in B_\nu} \| [\varphi_{lj}(z)] \|_{M_n} \left\| \sum_{j=1}^n \varepsilon_j \otimes x_j \right\|_{\text{Rad}(X)}$$

for any $n \geq 1$, for any matrix $[\varphi_{lj}]$ of elements of $H_0^\infty(B_\nu)$ and for any x_1, \dots, x_n in X .

Proof. Let us explain (1). We consider an $n \times n$ matrix $[f_{lj}]$ of elements of $H_0^\infty(\Sigma_\nu)$ and we associate $F \in H^\infty(\Sigma_\nu; M_n)$ defined by

$$F(z) = [f_{lj}(z)], \quad z \in \Sigma_\nu.$$

Then arguing as in the proof of Theorem 6.3 and applying the Franks-McIntosh decomposition principle with $Z = M_n$, we find a sequence $(b_k)_{k \geq 1}$ of $n \times n$ matrices $b_k = [b_k(l, j)]_{1 \leq l, j \leq n}$

such that

$$f_{lj}(z) = \sum_{k=1}^{\infty} b_k(l, j) F_k(z) G_k(z), \quad z \in \Sigma_{\theta},$$

for any $l, j = 1, \dots, n$, and

$$\sup_k \|b_k\|_{M_n} \leq C \sup\{\|[f_{lj}(z)]\|_{M_n} : z \in \Sigma_{\nu}\}.$$

Using the above results in the place of (6.6) and (6.7), the estimate (6.9) in the place of (6.3), and arguing as in the proof of Theorem 6.3, we obtain (6.10). Details are left to the reader.

Part (2) can be deduced from part (1) in the same manner that Theorem 6.4 was deduced from Theorem 6.3. \square

Part (2) of the above proposition generalizes [33, Thm. 3.3], where this property is proved for (commutative) L^p -spaces.

Remark 6.7. Property (6.10) means that the homomorphism $H_0^{\infty}(\Sigma_{\nu}) \rightarrow B(X)$ induced by the functional calculus is matricially R -bounded in the sense of [26, Section 4]. Restricting this property to column matrices, we obtain the property proved in Theorem 6.3. On the other hand, restricting (6.10) to diagonal matrices, we find the following unpublished result of Kalton-Weis [25]: if A has a bounded $H^{\infty}(\Sigma_{\theta})$ functional calculus on X with property (α) , then for any $\nu \in (\theta, \pi)$, the functional calculus homomorphism $H_0^{\infty}(\Sigma_{\nu}) \rightarrow B(X)$ maps the unit ball of $H_0^{\infty}(\Sigma_{\nu})$ into an R -bounded subset of $B(X)$.

We now turn to noncommutative L^p -spaces. We give ourselves a semifinite von Neumann algebra M . Recall that for any $n \geq 1$, the L^p -space associated with the von Neumann algebra $M_n(M)$ can be identified with the vector space of $n \times n$ matrices with entries in $L^p(M)$. The proof of Proposition 6.9 below will rely on the noncommutative Khintchine inequalities stated in Section 3 and the following elementary estimate.

Lemma 6.8. *For any $n \geq 1$, for any family $(\alpha_{kl})_{1 \leq k, l \leq n}$ of complex numbers, for any $2 \leq p \leq \infty$, and for any x_1, \dots, x_n in $L^p(M)$, we have*

$$(6.12) \quad \left\| [\alpha_{kl} x_k]_{1 \leq k, l \leq n} \right\|_{L^p(M_n(M))} \leq \sup_k \left(\sum_l |\alpha_{kl}|^2 \right)^{\frac{1}{2}} \left\| \left(\sum_k x_k^* x_k \right)^{\frac{1}{2}} \right\|_{L^p(M)}$$

and

$$(6.13) \quad \left\| [\alpha_{lk} x_l]_{1 \leq k, l \leq n} \right\|_{L^p(M_n(M))} \leq \sup_k \left(\sum_l |\alpha_{kl}|^2 \right)^{\frac{1}{2}} \left\| \left(\sum_l x_l x_l^* \right)^{\frac{1}{2}} \right\|_{L^p(M)}.$$

Proof. Let us prove (6.12) in the case $p = \infty$. Let $A = [\alpha_{kl} x_k]_{k, l \geq 1} \in M_n(M)$. Its adjoint is $A^* = [\overline{\alpha_{lk}} x_l^*]_{k, l \geq 1}$ hence the (k, l) -entry of the product $A^* A$ is equal to

$$\sum_j \overline{\alpha_{jk}} x_j^* \alpha_{jl} x_j.$$

Thus

$$A^*A = \sum_j c_j \otimes x_j^* x_j,$$

where $c_j = [\overline{\alpha_{jk}} \alpha_{jl}]_{k,l \geq 1} \in M_n$. Let I_n be the unit of M_n . Clearly each c_j is a nonnegative rank one matrix satisfying

$$0 \leq c_j \leq \left(\sum_l |\alpha_{jl}|^2 \right) I_n.$$

Therefore

$$A^*A \leq \left(\sup_k \sum_l |\alpha_{kl}|^2 \right) \left(\sum_j I_n \otimes x_j^* x_j \right).$$

Taking square roots yields (6.12).

That (6.12) holds in the case $p = 2$ is clear. Indeed,

$$\left\| \left(\sum_k x_k^* x_k \right)^{\frac{1}{2}} \right\|_{L^2(M)} = \left(\sum_k \|x_k\|_2^2 \right)^{\frac{1}{2}}$$

and we have

$$\left\| [\alpha_{kl} x_k] \right\|_{L^2(M_n(M))}^2 = \sum_{k,l} \|\alpha_{kl} x_k\|_2^2 = \sum_k \left(\sum_l |\alpha_{kl}|^2 \right) \|x_k\|_2^2 \leq \left(\sup_k \sum_l |\alpha_{kl}|^2 \right) \sum_k \|x_k\|_2^2.$$

Now (6.12) follows for any $2 \leq p \leq \infty$ by interpolation.

The proof of (6.13) is similar. □

Proposition 6.9. *For any $1 \leq p < \infty$, the space $L^p(M)$ has property (Q).*

Proof. Let $(\alpha_{kl})_{1 \leq k,l \leq n}$ be a family of complex numbers such that

$$(6.14) \quad \sum_{l=1}^n |\alpha_{kl}|^2 \leq 1, \quad k = 1, \dots, n.$$

Let x_1, \dots, x_n in $L^p(M)$.

Assume first that $2 \leq p < \infty$. By (6.14), we have

$$\sum_{k,l} (\alpha_{kl} x_k)^* (\alpha_{kl} x_k) = \sum_k \left(\sum_l |\alpha_{kl}|^2 \right) x_k^* x_k \leq \sum_{k=1}^n x_k^* x_k.$$

By (3.5), this implies

$$\left\| \left(\sum_{k,l} (\alpha_{kl} x_k)^* (\alpha_{kl} x_k) \right)^{\frac{1}{2}} \right\|_{L^p(M)} \lesssim \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(L^p(M))}.$$

Likewise,

$$\left\| \left(\sum_{k,l} (\alpha_{kl} x_k) (\alpha_{kl} x_k)^* \right)^{\frac{1}{2}} \right\|_{L^p(M)} \lesssim \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(L^p(M))}.$$

Moreover Lemma 6.8 ensures that

$$\max \left\{ \left\| [\alpha_{kl} x_k]_{k,l \geq 1} \right\|_{L^p(M_n(M))}, \left\| [\alpha_{lk} x_l]_{k,l \geq 1} \right\|_{L^p(M_n(M))} \right\} \lesssim \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(L^p(M))}.$$

According to the two-variable noncommutative Khintchine inequality (3.7), this yields (6.3).

Assume now that $1 \leq p \leq 2$. Let $x'_1, \dots, x'_n, x''_1, \dots, x''_n$ in $L^p(M)$ such that

$$x_k = x'_k + x''_k, \quad k = 1, \dots, n.$$

This decomposition immediately yields the following one:

$$\alpha_{kl}x_k = \alpha_{kl}x'_k + \alpha_{kl}x''_k, \quad k, l = 1, \dots, n.$$

Arguing as above we both have

$$\left\| \left(\sum_{k,l} (\alpha_{kl}x'_k)^* (\alpha_{kl}x'_k) \right)^{\frac{1}{2}} \right\|_{L^p(M)} \leq \left\| \left(\sum_k x_k'^* x_k' \right)^{\frac{1}{2}} \right\|_{L^p(M)}$$

and

$$\left\| \left(\sum_{k,l} (\alpha_{kl}x''_k) (\alpha_{kl}x''_k)^* \right)^{\frac{1}{2}} \right\|_{L^p(M)} \leq \left\| \left(\sum_k x_k'' x_k''^* \right)^{\frac{1}{2}} \right\|_{L^p(M)}.$$

Then combining (3.6) and (3.8), we deduce the estimate (6.3). \square

We now show going to show that Theorem 6.3 does not hold true on all Banach spaces, namely the next proposition shows that it fails on c_0 . Therefore c_0 does not have property (Q). A similar construction shows that Theorem 6.4 also fails on c_0 .

Proposition 6.10. *Let $A: c_0 \rightarrow c_0$ be defined by*

$$A(w) = (2^{-j}w_j)_{j \geq 1}, \quad w = (w_j)_{j \geq 1} \in c_0.$$

Then A is a sectorial operator and for any $\theta \in (0, \pi)$:

- (1) *A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*
- (2) *A does not have a quadratic $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof. The facts that A is sectorial and that property (1) holds are easy. Indeed, for any $\theta \in (0, \pi)$ and any $f \in H_0^\infty(\Sigma_\theta)$, we have

$$[f(A)](w) = (f(2^{-j})w_j)_{j \geq 1}$$

for any $w = (w_j)_{j \geq 1}$ in c_0 , and hence

$$\|f(A)\| \leq \|f\|_{L^\infty(0, \infty)}.$$

To prove (2), let us assume that A admits a quadratic $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \pi)$. Let $(e_j)_{j \geq 1}$ denote the canonical basis of c_0 . For any integers $n, m \geq 1$, for any w_1, \dots, w_m in \mathbb{C} and any f_1, \dots, f_n in $H_0^\infty(\Sigma_\theta)$,

$$\sum_l \varepsilon_l \otimes f_l(A) \left(\sum_j w_j e_j \right) = \sum_{l,j} f_l(2^{-j}) w_j \varepsilon_l \otimes e_j.$$

Hence there is a constant $C \geq 0$ (not depending on n, m, w_j) such that

$$\left\| \sum_{l=1}^n \sum_{j=1}^m f_l(2^{-j}) w_j \varepsilon_l \otimes e_j \right\|_{\text{Rad}(c_0)} \leq C \sup_j |w_j| \left\| \left(\sum_{l=1}^n |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_\theta}$$

for any f_1, \dots, f_n in $H_0^\infty(\Sigma_\theta)$, By an entirely classical approximation argument, the above estimate holds as well when the f_l 's belong to $H^\infty(\Sigma_\theta)$. Applying this with $w_j = 1$ for all j , one obtains

$$(6.15) \quad \left\| \sum_{l=1}^n \sum_{j=1}^m f_l(2^{-j}) \varepsilon_l \otimes e_j \right\|_{\text{Rad}(c_0)} \leq C \left\| \left(\sum_{l=1}^n |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_\theta}, \quad f_1, \dots, f_n \in H^\infty(\Sigma_\theta).$$

Let $Q_{n,m}: H^\infty(\Sigma_\theta; \ell_n^2) \rightarrow \ell_m^\infty(\ell_n^2)$ be defined by $Q_{n,m}(F) = (F(2^{-j}))_{1 \leq j \leq m}$. Then Q_n is onto and the vectorial form of Carleson's Interpolation Theorem (see [16, VII.2]) ensures that its lifting constant is bounded by a universal constant not depending on either m or n . Thus there is a constant $K \geq 1$ such that for any family $(\alpha_{lj})_{1 \leq j \leq m, 1 \leq l \leq n}$ of complex numbers there exists f_1, \dots, f_n in $H^\infty(\Sigma_\theta)$ such that

$$\left\| \left(\sum_{l=1}^n |f_l|^2 \right)^{\frac{1}{2}} \right\|_{\infty, \Sigma_\theta} \leq K \sup_{1 \leq j \leq m} \left(\sum_{l=1}^n |\alpha_{lj}|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \alpha_{lj} = f_l(2^{-j})$$

for any $1 \leq j \leq m, 1 \leq l \leq n$. It therefore follows from (6.15) that

$$(6.16) \quad \left\| \sum_{l=1}^n \sum_{j=1}^m \alpha_{lj} \varepsilon_l \otimes e_j \right\|_{\text{Rad}(c_0)} \leq CK \sup_{1 \leq j \leq m} \left(\sum_{l=1}^n |\alpha_{lj}|^2 \right)^{\frac{1}{2}}, \quad \alpha_{lj} \in \mathbb{C}.$$

Let $n \geq 1$ be an integer. Since the unit ball of ℓ_n^2 is compact, there exists a finite family (y_1, \dots, y_m) of that unit ball such that

$$(6.17) \quad \|y\|_{\ell_n^2} \leq 2 \sup \{ |\langle y, y_j \rangle| : j = 1, \dots, m \}$$

for any $y \in \ell_n^2$. Let (h_1, \dots, h_n) be an orthonormal basis of ℓ_n^2 , and let

$$\alpha_{lj} = \langle h_l, y_j \rangle, \quad 1 \leq j \leq m, 1 \leq l \leq n.$$

Then the supremum in the right handside of (6.16) is equal to $\sup_j \|y_j\|$, hence is less than or equal to 1. Consequently,

$$\left\| \sum_{l=1}^n \sum_{j=1}^m \langle h_l, y_j \rangle \varepsilon_l \otimes e_j \right\|_{\text{Rad}(c_0)} \leq CK.$$

Now observe that for any $u \in \mathcal{M}$,

$$\begin{aligned} \left\| \sum_{l=1}^n \sum_{j=1}^m \langle h_l, y_j \rangle \varepsilon_l(u) e_j \right\|_{c_0} &= \left\| \sum_{j=1}^m \left\langle \sum_{l=1}^n \varepsilon_l(u) h_l, y_j \right\rangle e_j \right\|_{c_0} \\ &= \sup_j \left| \left\langle \sum_{l=1}^n \varepsilon_l(u) h_l, y_j \right\rangle \right|. \end{aligned}$$

Since (h_1, \dots, h_n) is an orthonormal basis, the norm of $\sum_{l=1}^n \varepsilon_l(u) h_l$ in ℓ_n^2 is equal to $n^{\frac{1}{2}}$. Appling (6.17), we deduce that

$$n^{\frac{1}{2}} \leq 2 \left\| \sum_{l=1}^n \sum_{j=1}^m \langle h_l, y_j \rangle \varepsilon_l(u) e_j \right\|_{c_0}.$$

Integrating over \mathcal{M} , this yields $n^{\frac{1}{2}} \leq 2CK$ for any $n \geq 1$, a contradiction. \square

We now come back to the question addressed at the beginning of this section. The following classical result will be used in the next proof: If a Banach space X does not contain c_0 (as an isomorphic subspace), then a series $\sum_k \varepsilon_k \otimes x_k$ converges in $L^2(\mathcal{M}; X)$ if and only if its partial sums are uniformly bounded (see [28]).

Proposition 6.11. *Assume that X does not contain c_0 . Let $T: X \rightarrow X$ be a Ritt operator and assume that T has a quadratic $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$. Then for any $m \geq 1$, T satisfies a uniform estimate*

$$(6.18) \quad \|x\|_{T,m} \lesssim \|x\|, \quad x \in X.$$

Proof. According to the property discussed before the statement, it suffices to show the existence of a constant $K > 0$ such that for any $n \geq 1$ and any $x \in X$,

$$\left\| \sum_{l=1}^n l^{m-\frac{1}{2}} \varepsilon_l \otimes T^{l-1}(I-T)^m x \right\|_{\text{Rad}(X)} \leq K \|x\|.$$

This is obtained by applying Definition 6.1, (2), with

$$\varphi_l(z) = l^{m-\frac{1}{2}} z^{l-1} (1-z)^l,$$

see the proof of [33, Thm. 3.3] for the details. \square

Let us finally summarize what we obtain by combining Theorem 6.4, and Propositions 6.11, 6.5 and 6.9. Recall that by Proposition 6.10, a Banach space with property (Q) cannot contain c_0 .

Corollary 6.12.

- (1) *Assume that X has property (Q). Let $T: X \rightarrow X$ be a Ritt operator with a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$. Then it satisfies a square function estimate (6.18) for any $m \geq 1$.*
- (2) *Part (1) applies to Banach spaces with property (α) and to noncommutative L^p -spaces for $1 \leq p < \infty$.*

7. FROM SQUARE FUNCTIONS TO H^∞ FUNCTIONAL CALCULUS

This section is devoted to the issue of showing that a Ritt operator has a bounded H^∞ -functional calculus with respect to a Stolz domain B_γ , provided that it satisfies suitable square function estimates. We consider an arbitrary Banach space X and first establish a general result, namely Theorem 7.3 below. Then we consider special cases in the last part of the section.

Lemma 7.1. *Let $0 < \alpha < \gamma < \frac{\pi}{2}$ and let $T: X \rightarrow X$ be a Ritt operator of type α (resp. an R -Ritt operator of R -type α). There exists a constant $C > 0$ such that for any $\varphi \in H_0^\infty(B_\gamma)$, we have*

$$k \|\varphi(T)(T^k - T^{k-1})\| \leq C \|\varphi\|_{\infty, B_\gamma}, \quad k \geq 1$$

(resp. the set $\{k\varphi(T)(T^k - T^{k-1}) : k \geq 1\}$ is R -bounded and

$$\mathcal{R}\left(\{k\varphi(T)(T^k - T^{k-1}) : k \geq 1\}\right) \leq C\|\varphi\|_{\infty, B_\gamma}.$$

Proof. We will prove this result in the ‘ R -Ritt case’ only, the ‘Ritt case’ being similar and simpler. We fix a real number $\beta \in (\alpha, \gamma)$. Recall Lemma 5.2 and let

$$C_1 = \mathcal{R}\left(\{(\lambda - 1)R(\lambda, T) : \lambda \in \partial B_\beta \setminus \{1\}\}\right).$$

For any function $\varphi \in H_0^\infty(B_\gamma)$ and any integer $k \geq 1$, we have

$$k\varphi(T)(T^k - T^{k-1}) = \frac{1}{2\pi i} \int_{\partial B_\beta} k\varphi(\lambda)\lambda^{k-1}((\lambda - 1)R(\lambda, T)) d\lambda.$$

Hence by Lemma 5.1, we have

$$\begin{aligned} \mathcal{R}\left(\{k\varphi(T)(T^k - T^{k-1}) : k \geq 1\}\right) &\leq \frac{C_1}{\pi} \sup_{k \geq 1} \left\{ k \int_{\partial B_\beta} |\varphi(\lambda)| |\lambda|^{k-1} |d\lambda| \right\} \\ &\leq \frac{C_1}{\pi} \|\varphi\|_{\infty, B_\gamma} \sup_{k \geq 1} \left\{ k \int_{\partial B_\beta} |\lambda|^{k-1} |d\lambda| \right\}. \end{aligned}$$

The finiteness of the latter supremum is well-known, see e.g. [52, Lem. 2.1] and its proof. The result follows at once. \square

Lemma 7.2. *Let $T : X \rightarrow X$ be a Ritt operator. For any $x \in \overline{\text{Ran}(I - T)}$, we have*

$$\sum_{k=1}^{\infty} k(k+1)T^{k-1}(I - T)^3 x = 2x.$$

Proof. Let $N \geq 1$ be an integer. First, we have

$$\begin{aligned} \sum_{k=1}^N k(k+1)T^{k-1}(I - T) &= \sum_{k=1}^N k(k+1)T^{k-1} - \sum_{k=2}^{N+1} (k-1)kT^{k-1} \\ &= 2 \sum_{k=1}^N kT^{k-1} - N(N+1)T^N. \end{aligned}$$

Then we compute

$$\sum_{k=1}^N kT^{k-1}(I - T) = \sum_{k=1}^N kT^{k-1} - \sum_{k=2}^{N+1} (k-1)T^{k-1} = \sum_{k=1}^N T^{k-1} - NT^N,$$

and we note that

$$\sum_{k=1}^N T^{k-1}(I - T) = I - T^N.$$

Putting these identities together, we obtain that

$$(7.1) \quad \sum_{k=1}^N k(k+1)T^{k-1}(I - T)^3 = 2I - 2T^N - 2NT^N(I - T) - N(N+1)T^N(I - T)^2.$$

Since T is a Ritt operator, the four sequences

$$\mathcal{S}_0 = (T^N)_{N \geq 1}, \quad \mathcal{S}_1 = (NT^N(I - T))_{N \geq 1}, \quad \mathcal{S}_2 = (N^2T^N(I - T)^2)_{N \geq 1}$$

and

$$\mathcal{S}_3 = (N^3T^N(I - T)^3)_{N \geq 1}$$

are bounded (see [52, Lem. 2.1]).

If $x = (I - T)z$ is an element of $\text{Ran}(I - T)$, the boundedness of \mathcal{S}_3 implies that

$$N(N + 1)T^N(I - T)^2x = \frac{N + 1}{N^2} N^3T^N(I - T)^3z \longrightarrow 0$$

when $N \rightarrow \infty$. Then the boundedness of the sequence \mathcal{S}_2 implies that we actually have

$$\lim_N N(N + 1)T^N(I - T)^2x = 0$$

for any x in the closure $\overline{\text{Ran}(I - T)}$. Likewise, using $\mathcal{S}_2, \mathcal{S}_1$ and \mathcal{S}_0 , we have

$$\lim_N NT^N(I - T)x = 0 \quad \text{and} \quad \lim_N T^Nx = 0$$

for any $x \in \overline{\text{Ran}(I - T)}$. Thus applying (7.1) yields the result. \square

Theorem 7.3. *Let $T: X \rightarrow X$ be an R -Ritt operator of R -type $\alpha \in (0, \frac{\pi}{2})$. If T and T^* both satisfy uniform estimates*

$$\|x\|_{T,1} \lesssim \|x\| \quad \text{and} \quad \|y\|_{T^*,1} \lesssim \|y\|$$

for $x \in X$ and $y \in X^*$, then T admits a bounded $H^\infty(B_\gamma)$ functional calculus for any $\gamma \in (\alpha, \frac{\pi}{2})$.

Proof. We fix γ in $(\alpha, \frac{\pi}{2})$. Let $\omega = e^{\frac{2i\pi}{3}}$, then the two operators $\omega I - T$ and $\bar{\omega}I - T$ are invertible and $I - T^3 = (I - T)(\omega I - T)(\bar{\omega}I - T)$. Hence

$$\text{Ran}(I - T) = \text{Ran}(I - T^3).$$

Note moreover that T^3 is a Ritt operator.

Let $\varphi \in \mathcal{P}$ such that $\varphi(1) = 0$ and consider $x \in X$. Then $\varphi(T)x \in \text{Ran}(I - T)$ hence applying Lemma 7.2 to T^3 and the above observations, we obtain

$$\sum_{k=1}^{\infty} k(k+1)T^{3(k-1)}(I - T^3)^3\varphi(T)x = 2\varphi(T)x.$$

We set $\psi(T) = (I + T + T^2)^3/2$ for convenience, so that $2\psi(T)(I - T)^3 = (I - T^3)^3$. Then for any $y \in X^*$, we derive

$$\begin{aligned} \langle \varphi(T)x, y \rangle &= \sum_{k=1}^{\infty} \langle k(k+1)\psi(T)\varphi(T)T^{3(k-1)}(I - T)^3x, y \rangle \\ &= \sum_{k=1}^{\infty} \langle [(k+1)\varphi(T)T^{k-1}(I - T)]k^{\frac{1}{2}}T^{k-1}(I - T)x, k^{\frac{1}{2}}T^{*(k-1)}(I - T^*)\psi(T^*)y \rangle. \end{aligned}$$

Note that for any finite families $(x_k)_{k \geq 1}$ in X and $(y_k)_{k \geq 1}$ in X^* , we have

$$\sum_k \langle x_k, y_k \rangle = \int_{\mathcal{M}} \left\langle \sum_k \varepsilon_k(u) x_k, \sum_k \varepsilon_k(u) y_k \right\rangle d\mathbb{P}(u),$$

and hence

$$\left| \sum_k \langle x_k, y_k \rangle \right| \leq \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)} \left\| \sum_k \varepsilon_k \otimes y_k \right\|_{\text{Rad}(X^*)}$$

by Cauchy-Schwarz.

Thus for any integer $N \geq 1$, we have

$$\begin{aligned} & \left| \sum_{k=1}^N \left\langle [(k+1)\varphi(T)T^{k-1}(I-T)] k^{\frac{1}{2}} T^{k-1}(I-T)x, k^{\frac{1}{2}} T^{*(k-1)}(I-T^*)\psi(T^*)y \right\rangle \right| \\ & \leq \left\| \sum_{k=1}^N \varepsilon_k \otimes [(k+1)\varphi(T)T^{k-1}(I-T)] k^{\frac{1}{2}} T^{k-1}(I-T)x \right\|_{\text{Rad}(X)} \\ & \quad \times \left\| \sum_{k=1}^N \varepsilon_k \otimes k^{\frac{1}{2}} T^{*(k-1)}(I-T^*)\psi(T^*)y \right\|_{\text{Rad}(X^*)} \\ & \leq \mathcal{R}\left(\{(k+1)\varphi(T)T^{k-1}(I-T) : k \geq 1\}\right) \|\psi(T)\| \|x\|_{T,1} \|y\|_{T^*,1} \\ & \lesssim \|\varphi\|_{\infty, B_\gamma} \|x\|_{T,1} \|\psi(T^*)y\|_{T^*,1} \end{aligned}$$

by Lemma 7.1. Applying our assumptions, we deduce that

$$|\langle \varphi(T)x, y \rangle| \lesssim \|\varphi\|_{\infty, B_\gamma} \|x\| \|y\|.$$

Since x and y are arbitrary, this implies an estimate $\|\varphi(T)\| \lesssim \|\varphi\|_{\infty, B_\gamma}$ for polynomials vanishing at 1. Writing any polynomial as $\varphi = \varphi(1) + (\varphi - \varphi(1))$, we immediatly derive a similar estimate for all polynomials. This yields the result by Proposition 2.5. \square

Theorem 7.3 fails if we remove one of the two square function estimates in the assumption. This will follow from Proposition 8.2.

We finally consider special cases and combinations with results from the previous sections. Following [24], we say that a Banach space X has property (Δ) if the triangular projection is bounded on $\text{Rad}(\text{Rad}(X))$, that is, there exists a constant $C > 0$ such that for finite doubly indexed families $(x_{kl})_{k,l \geq 1}$ of X ,

$$\left\| \sum_{k \geq 1} \sum_{l \geq k} \varepsilon_k \otimes \varepsilon_l \otimes x_{kl} \right\|_{\text{Rad}(\text{Rad}(X))} \leq C \left\| \sum_{k \geq 1} \sum_{l \geq 1} \varepsilon_k \otimes \varepsilon_l \otimes x_{kl} \right\|_{\text{Rad}(\text{Rad}(X))}.$$

That condition is clearly weaker than (α) . Furthermore any UMD Banach space has property (Δ) , by [24, Prop. 3.2]. Thus any noncommutative L^p -space with $1 < p < \infty$ has property (Δ) .

It follows from [24, Thm. 5.3] that if A is a sectorial operator on X with property (Δ) and A admits a bounded $H^\infty(\Sigma_\theta)$ for some $\theta < \frac{\pi}{2}$, then A is R -sectorial of R -type $< \frac{\pi}{2}$. Combining with (the easy implication of) Proposition 4.1 and Lemma 5.2, we deduce the following.

Proposition 7.4. *Let T be a Ritt operator on X with property (Δ) . If T admits a bounded $H^\infty(B_\gamma)$ for some $\gamma < \frac{\pi}{2}$, then T is R -Ritt.*

Combining further with Corollary 6.12, we obtain the following equivalence result.

Corollary 7.5.

- (1) *Assume that X has both properties (Q) and (Δ) and let $T: X \rightarrow X$ be a Ritt operator. The following assertions are equivalent.*
- (i) *T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*
 - (ii) *T is R -Ritt and T and T^* both satisfy uniform estimates*

$$\|x\|_{T,1} \lesssim \|x\| \quad \text{and} \quad \|y\|_{T^*,1} \lesssim \|y\|$$

for $x \in X$ and $y \in X^$*

- (2) *Part (1) applies to Banach spaces with property (α) and to noncommutative L^p -spaces for $1 < p < \infty$.*

If $X = L^p(\Omega)$ is a commutative L^p -space with $1 < p < \infty$, then demanding that T is R -Ritt in condition (ii) is superfluous, by Theorem 5.3. In this case, the above statement yields Theorem 1.1. According to this discussion and Remark 5.5, Theorem 1.1 holds true as well on any reflexive space with property (α) .

We conclude this section with an observation of independent interest on the role of the R -Ritt condition in the study of $H^\infty(B_\gamma)$ functional calculus. Recall Definition 2.6.

Proposition 7.6. *Let $T: X \rightarrow X$ be an R -Ritt operator of R -type α . If T is polynomially bounded, then it admits a bounded $H^\infty(B_\gamma)$ functional calculus for any $\gamma \in (\alpha, \frac{\pi}{2})$.*

Proof. As was observed in Section 5, the operator $A = I - T$ is R -sectorial of R -type α . Moreover the proof of the easy implication ‘(i) \Rightarrow (ii)’ of Proposition 4.1 shows that A admits a bounded $H^\infty(\Sigma_{\frac{\pi}{2}})$ functional calculus. According to [24, Prop. 5.1], this implies that for any $\theta \in (\alpha, \pi)$, A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. The result therefore follows from Proposition 4.1. \square

8. EXAMPLES AND ILLUSTRATIONS

In this final section, we give complements for the following 3 classes of Banach spaces: Hilbert spaces, commutative L^p -spaces, noncommutative L^p -spaces. We give either characterizations of Ritt operators satisfying the equivalent conditions of Corollary 7.5, or exhibit classes of examples satisfying these conditions.

8.a. Hilbert spaces. Let H be a Hilbert space. Two bounded operators $S, T: H \rightarrow H$ are called similar provided that there is an invertible operator $V \in B(H)$ such that $S = V^{-1}TV$. In particular we say that T is similar to a contraction if there is an invertible operator $V \in B(H)$ such that $\|V^{-1}TV\| \leq 1$. This is equivalent to the existence of an equivalent Hilbertian norm on H with respect to which T is contractive. Any T similar to a contraction is polynomially bounded (by von Neumann’s inequality). Pisier’s negative solution to the Halmos problem asserts that the converse is wrong, see [46] for details and complements on similarity problems. It is known however that any Ritt operator which is polynomially

bounded is necessarily similar to a contraction, see [30, 12]. The next statement (which may be known to some similarity specialists) is a refinement of that result, also containing the Hilbert space version of Theorem 1.1.

Note that the class of Ritt operators is stable under similarity.

Theorem 8.1. *For any power bounded operator $T \in B(H)$, the following assertions are equivalent.*

- (i) *T is a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*
- (ii) *T and T^* both satisfy uniform estimates*

$$\|x\|_{T,1} \lesssim \|x\| \quad \text{and} \quad \|y\|_{T^*,1} \lesssim \|y\|$$

for $x, y \in H$.

- (iii) *T is a Ritt operator and T is similar to a contraction.*

Proof. It follows from [23, Thm. 4.7] that T is a Ritt operator if it satisfies (ii). With this result in hands, the equivalence between (i) and (ii) reduces to Corollary 7.5.

If T satisfies (iii), then it is polynomially bounded (see the discussion above). Hence it satisfies (i) by Proposition 7.6.

Assume (ii). Recall (5.6) (with $X = H$) and let $P: H \rightarrow H$ be the projection onto $\text{Ker}(I - T)$ whose kernel equals $\overline{\text{Ran}(I - T)}$. Then we have an equivalence

$$(8.1) \quad \|x\| \approx (\|P(x)\|^2 + \|x\|_{T,1}^2)^{\frac{1}{2}}, \quad x \in H.$$

Indeed this follows from the proof of [23, Thm. 4.7], see also [33, Cor. 3.4]. Let $|||x|||$ denote the right handside of (8.1). Then $|||\cdot|||$ is an equivalent Hilbertian norm on H . Further for any $x \in H$,

$$\|T(x)\|_{T,1}^2 = \sum_{k=1}^{\infty} k \|T^{k+1}(x) - T^k(x)\|^2 \leq \sum_{k=2}^{\infty} k \|T^k(x) - T^{k-1}(x)\|^2 \leq \|x\|_{T,1}^2.$$

This implies that T is a contraction on $(H, |||\cdot|||)$. Thus T is similar to a contraction, which shows (iii). \square

A natural question (also making sense on general Banach spaces) is whether one can get rid of one of the two square function estimates of (ii) in the above equivalence result. It turns out that the answer is negative.

Proposition 8.2. *There exists a Ritt operator T on Hilbert space which is not similar to a contraction, although it satisfies an estimate*

$$\|x\|_{T,1} \lesssim \|x\|, \quad x \in H.$$

Proof. This is a simple adaptation of [31, Thm. 5.2] so we will be brief. Let H be a separable infinite dimensional Hilbert space and let $(e_m)_{m \geq 1}$ be a normalized Schauder basis on H which satisfies an estimate

$$(8.2) \quad \left(\sum_m |t_m|^2 \right)^{\frac{1}{2}} \lesssim \left\| \sum_m t_m e_m \right\|$$

for finite sequences $(t_m)_{m \geq 1}$ of complex numbers but for which there is no reverse estimate, that is,

$$(8.3) \quad \sup \left\{ \left\| \sum_m t_m e_m \right\| : \sum_m |t_m|^2 \leq 1 \right\} = \infty.$$

Let $T: H \rightarrow H$ be defined by letting

$$T \left(\sum_m t_m e_m \right) = \sum_m (1 - 2^{-m}) t_m e_m.$$

According to e.g. [29, Thm. 4.1], this operator is well-defined and $A = I - T$ is sectorial of any positive type. Moreover $\sigma(T) \subset [0, 1]$, hence T is a Ritt operator.

Arguing as in the proof of [31, Thm. 5.2], one obtains an equivalence

$$\left\| \sum_m t_m e_m \right\|_{T,1} \approx \left(\sum_m |t_m|^2 \right)^{\frac{1}{2}}$$

for finite sequences $(t_m)_{m \geq 1}$ of complex numbers.

In view of (8.2), this implies the square function estimate $\|x\|_{T,1} \lesssim \|x\|$. If T were similar to a contraction, it would satisfy an estimate $\|y\|_{T^*,1} \lesssim \|y\|$, by Theorem 8.1. It would therefore satisfy a reverse estimate $\|x\| \lesssim \|x\|_{T,1}$ by (8.1). This contradicts (8.3). \square

8.b. Commutative L^p -spaces. Let (Ω, μ) be a measure space and let $1 < p < \infty$. The following is the main result of [33], we provide a proof in the light of the present paper.

Theorem 8.3. [33] *Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a positive contraction and assume that T is a Ritt operator. Then it satisfies the equivalent conditions of Theorem 1.1.*

Proof. Let $(T_t)_{t \geq 0}$ be the uniformly continuous semigroup on $L^p(\Omega)$ defined by

$$T_t = e^{-t} e^{tT}, \quad t \geq 0.$$

Then for any $t \geq 0$, T_t is positive and $\|T_t\| \leq e^{-t} e^{t\|T\|} \leq 1$. The generator of $(T_t)_{t \geq 0}$ is $T - I = -A$ and since T is a Ritt operator, A is sectorial of type $< \frac{\pi}{2}$. Hence A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{\pi}{2}$, by [33, Prop. 2.2]. According to Proposition 4.1, this implies condition (i) of Theorem 1.1. \square

For applications of this result to ergodic theory, see [34].

Ritt operators on $L^p(\Omega)$ satisfying Theorem 1.1 do not have any description comparable to the one given by Theorem 8.1 on Hilbert space. However in a separate joint work with C. Arhancet [4], we show that for an R -Ritt operator $T: L^p(\Omega) \rightarrow L^p(\Omega)$, T satisfies the conditions of Theorem 1.1 if and only if there exists a second measure space (Ω', μ') , two bounded maps $J: L^p(\Omega) \rightarrow L^p(\Omega')$ and $Q: L^p(\Omega') \rightarrow L^p(\Omega)$, as well as an isomorphism $U: L^p(\Omega') \rightarrow L^p(\Omega')$ such that $\{U^n : n \in \mathbb{Z}\}$ is bounded and

$$T^n = QU^n J, \quad n \geq 0.$$

8.c. Noncommutative L^p -spaces. In this subsection, we let M be a semifinite von Neumann algebra equipped with a semifinite faithful trace τ . Thanks to the noncommutative Khintchine inequalities (3.5) and (3.6), Corollary 7.5 has a specific form on $L^p(M)$. We state it in

the case $2 \leq p < \infty$, the dual case ($1 < p \leq 2$) can be obtained by changing T into T^* . This is the noncommutative analog of Theorem 1.1, the square functions (1.3) being replaced by their natural noncommutative versions.

Corollary 8.4. *Let $2 \leq p < \infty$ and let $T: L^p(M) \rightarrow L^p(M)$ be a Ritt operator. Then T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{\pi}{2}$ if and only if T is R-Ritt and there exists a constant $C > 0$ such that the following three estimates hold:*

(1) *For any $x \in L^p(M)$,*

$$\left\| \left(\sum_{k=1}^{\infty} k |T^k(x) - T^{k-1}(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \leq C \|x\|_{L^p(M)}.$$

(2) *For any $x \in L^p(M)$,*

$$\left\| \left(\sum_{k=1}^{\infty} k |(T^k(x) - T^{k-1}(x))^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \leq C \|x\|_{L^p(M)}.$$

(3) *For any $y \in L^{p'}(M)$, there exists two sequences $(u_k)_{k \geq 1}$ and $(v_k)_{k \geq 1}$ in $L^{p'}(M)$ such that*

$$k^{\frac{1}{2}} (T^{*k}(y) - T^{*(k-1)}(y)) = u_k + v_k$$

for any $k \geq 1$, and

$$\left\| \left(\sum_{k=1}^{\infty} |u_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(M)} \leq C \|y\|_{L^{p'}(M)} \quad \text{and} \quad \left\| \left(\sum_{k=1}^{\infty} |v_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(M)} \leq C \|y\|_{L^{p'}(M)}.$$

We will now exhibit two classes of examples satisfying the conditions of the above corollary. We start with Schur multipliers. Here our von Neumann algebra is $B(\ell^2)$, the trace τ is the usual trace and the associated noncommutative L^p -spaces are the Schatten classes that we denote by S^p . We represent any element of $B(\ell^2)$ by a bi-infinite matrix in the usual way. We recall that a bounded Schur multiplier on $B(\ell^2)$ is a bounded map $T: B(\ell^2) \rightarrow B(\ell^2)$ of the form

$$(8.4) \quad [c_{ij}]_{i,j \geq 1} \xrightarrow{T} [t_{ij} c_{ij}]_{i,j \geq 1}$$

for some matrix $[t_{ij}]_{i,j \geq 1}$ of complex numbers. See e.g. [46, Thm 5.1] for a description of those maps. It is well-known (using duality and interpolation) that any bounded Schur multiplier $T: B(\ell^2) \rightarrow B(\ell^2)$ extends to a bounded map $T: S^p \rightarrow S^p$ for any $1 \leq p < \infty$, with

$$\|T: S^p \rightarrow S^p\| \leq \|T: B(\ell^2) \rightarrow B(\ell^2)\|.$$

In particular, any contractive Schur multiplier $T: B(\ell^2) \rightarrow B(\ell^2)$ extends to a contraction on S^p for any p . In this case, the complex numbers t_{ij} given by (8.4) have modulus ≤ 1 . Moreover, $T: S^2 \rightarrow S^2$ is selfadjoint (in the usual Hilbertian sense) if and only if the associated matrix $[t_{ij}]_{i,j \geq 1}$ is real valued.

We say that a semigroup $(T_t)_{t \geq 0}$ of contractive Schur multipliers on $B(\ell^2)$ is w^* -continuous if $w^*\text{-}\lim_{t \rightarrow 0} T_t(x) = x$ for any $x \in B(\ell^2)$. In this case, $(T_t)_{t \geq 0}$ extends to a strongly continuous semigroup of S^p for any $1 \leq p < \infty$. Further we say that $(T_t)_{t \geq 0}$ is selfadjoint provided that $T_t: S^2 \rightarrow S^2$ is selfadjoint for any $t \geq 0$. See [21, Chapter 5] for the more general notion of noncommutative diffusion semigroup.

In the sequel we let $\omega_p = \pi \left| \frac{1}{p} - \frac{1}{2} \right|$. The following extends [21, 8.C].

Proposition 8.5. *Let $(T_t)_{t \geq 0}$ be a selfadjoint w^* -continuous semigroup of contractive Schur multipliers on $B(\ell^2)$. For any $1 < p < \infty$, let $-A_p$ be the infinitesimal generator of $(T_t)_{t \geq 0}$ on S^p . Then for any $\theta \in (\omega_p, \pi)$, A_p admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof. For any $1 < p < \infty$, let $(U_{t,p})_{t \in \mathbb{R}}$ be the translation semigroup on the Bochner space $L^p(\mathbb{R}; S^p)$. Then it follows from [3, Cor. 4.3 & Thm. 5.3] that for any $b \in L^1(0, \infty)$,

$$\left\| \int_0^\infty b(t) T_t dt : S^p \longrightarrow S^p \right\| \leq \left\| \int_0^\infty b(t) U_{t,p} dt : L^p(\mathbb{R}; S^p) \longrightarrow L^p(\mathbb{R}; S^p) \right\|.$$

Let C_p be the negative generator of $(U_{t,p})_{t \in \mathbb{R}}$. By [29, Lem. 2.12], the above inequality implies that for any $\theta > \frac{\pi}{2}$ and any $f \in H_0^\infty(\Sigma_\theta)$,

$$\|f(A_p)\| \leq \|f(C_p)\|.$$

Since S^p is a UMD Banach space, C_p has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$ (see e.g. [20]). Hence the above estimate implies that in turn, A_p has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$.

We assumed that $(T_t)_{t \geq 0}$ is selfadjoint. Hence by [21, Prop. 5.8], the above property holds true as well for any $\theta > \omega_p$. \square

Recall Definition 2.6 for polynomial boundedness.

Corollary 8.6. *Let $T: B(\ell^2) \rightarrow B(\ell^2)$ be a contractive Schur multiplier associated with a real valued matrix $[t_{ij}]_{i,j \geq 1}$ and let $1 < p < \infty$.*

- (1) *The induced operator $T: S^p \rightarrow S^p$ is polynomially bounded.*
- (2) *If there exists $\delta > 0$ such that $t_{ij} \geq -1 + \delta$ for any $i, j \geq 1$, then the induced operator $T: S^p \rightarrow S^p$ is a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{1}{2}$. When $p \geq 2$, it satisfies the conditions (1)-(2)-(3) of Corollary 8.4.*

Proof. We first prove (2). Assume that $t_{ij} \geq -1 + \delta$ for any $i, j \geq 1$. Then the spectrum of the selfadjoint map $T: S^2 \rightarrow S^2$ is included in $[-1 + \delta, 1]$. Applying the Spectral Theorem, this readily implies that $T: S^2 \rightarrow S^2$ is a Ritt operator. According to [33, Lem. 5.1], this implies that for any $1 < p < \infty$, $T: S^p \rightarrow S^p$ is a Ritt operator.

For any $t \geq 0$, $T_t = e^{-t} e^{tT}$ is a contractive selfadjoint Schur multiplier. Hence for any $1 < p < \infty$, $A = I - T$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on S^p for any $\theta > \omega_p$, by Proposition 8.5. Note that $\omega_p < \frac{\pi}{2}$. Thus the result now follows from Proposition 4.1 and Corollary 8.4.

We now prove (1). Under our assumption, the square operator $T^2: B(\ell^2) \rightarrow B(\ell^2)$ is a contractive Schur multiplier, and its associated matrix is $[t_{ij}^2]_{i,j \geq 1}$. Hence T^2 satisfies part (2) of the present corollary. Let $1 < p < \infty$. Since polynomial boundedness is implied by the existence of a bounded $H^\infty(B_\gamma)$ functional calculus, we deduce from (2) that there exists a constant $K_p \geq 1$ such that

$$\|\varphi(T^2)\|_{B(S^p)} \leq K_p \|\varphi\|_{\infty, \mathbb{D}}, \quad \varphi \in \mathcal{P}.$$

Any polynomial φ admits a (necessarily unique) decomposition

$$\varphi(z) = \varphi_1(z^2) + z\varphi_2(z^2)$$

and it is easy to check that

$$\|\varphi_1\|_{\infty, \mathbb{D}} \leq \|\varphi\|_{\infty, \mathbb{D}} \quad \text{and} \quad \|\varphi_2\|_{\infty, \mathbb{D}} \leq \|\varphi\|_{\infty, \mathbb{D}}.$$

Writing $\varphi(T) = \varphi_1(T^2) + T\varphi_2(T^2)$, we deduce that

$$\begin{aligned} \|\varphi(T)\|_{B(S^p)} &\leq \|\varphi_1(T^2)\|_{B(S^p)} + \|\varphi_2(T^2)\|_{B(S^p)} \\ &\leq K_p(\|\varphi_1\|_{\infty, \mathbb{D}} + \|\varphi_2\|_{\infty, \mathbb{D}}) \\ &\leq 2K_p\|\varphi\|_{\infty, \mathbb{D}}. \end{aligned}$$

□

We now turn to our second class of examples. Here we assume that τ is finite and normalized, that is, $\tau(1) = 1$. In this case, $M \subset L^p(M)$ for any $1 \leq p < \infty$. Following [17, 49], we say that a linear map $T: M \rightarrow M$ is a Markov map if T is unital, completely positive and trace preserving. As is well-known, such a map is necessarily normal and for any $1 \leq p < \infty$, it extends to a contraction $T_p: L^p(M) \rightarrow L^p(M)$. We say that T is selfadjoint if its L^2 -realization T_2 is selfadjoint in the usual Hilbertian sense.

Applying the techniques developed so far, the following analog of Corollary 8.6 is a rather direct consequence of some recent work of M. Junge, É. Ricard and D. Shlyakhtenko.

Proposition 8.7. *Let $T: M \rightarrow M$ be a selfadjoint Markov map.*

- (1) *For any $1 < p < \infty$, the operator $T_p: L^p(M) \rightarrow L^p(M)$ is polynomially bounded.*
- (2) *If $-1 \notin \sigma(T_2)$, then for any $1 < p < \infty$, $T_p: L^p(M) \rightarrow L^p(M)$ is a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{1}{2}$. When $p \geq 2$, it satisfies the conditions (1)-(2)-(3) of Corollary 8.4.*

Proof. Let $A_p = I_{L^p(M)} - T_p$ for any $1 < p < \infty$. Repeating the method applied to deduce Corollary 8.6 from Proposition 8.5, we see that it suffices to show that for any $1 < p < \infty$, A_p is sectorial and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{\pi}{2}$.

For that purpose, consider

$$T_t = e^{-t(I-T)}, \quad t \geq 0.$$

Then $(T_t)_{t \geq 0}$ is a ‘noncommutative diffusion semigroup’ in the sense of [21, Chapter 5], and for any $1 < p < \infty$, $-A_p$ is the generator of its L^p -realization. Hence A_p is sectorial by [21, Prop. 5.4].

According to [22], each T_t is ‘factorizable’ in the sense of [2, Def. 6.2] or [17, Def. 1.3]. Writing $T_t = T_{\frac{t}{2}}^2$ and using [17, Thm. 5.3] we deduce that each T_t satisfies the ‘Rota dilation property’ introduced in [21, Def. 10.2] (see also [17, Def. 5.1]).

We deduce the result by applying the reasoning in [21, 10.D]. Indeed it is implicitly shown there that whenever $(T_t)_{t \geq 0}$ is a diffusion semigroup on a finite von Neumann algebra such that each T_t satisfies the Rota dilation property, then the negative generator of its L^p -realization admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \omega_p$. □

Remark 8.8.

(1) About the necessity of having two parts in Proposition 8.7, we note that L^p -realizations of selfadjoint Markov maps are not necessarily Ritt operators. For instance, the mapping $T: \ell_2^\infty \rightarrow \ell_2^\infty$ defined by $T(t, s) = (s, t)$ is a Markov map but $-1 \in \sigma(T)$.

(2) If $T: M \rightarrow M$ satisfies the Rota dilation property, then it is a Markov map and its L^2 -realization is positive in the Hilbertian sense. Hence it satisfies Proposition 8.7. In this case, the latter statement strengthens [21, Cor. 10.9], where weaker square function estimates were established for operators with the Rota dilation property.

(3) For any selfadjoint Schur multiplier (resp. Markov map) T , the square operator T^2 satisfies the second part of Corollary 8.6 (resp. Proposition 8.7). Hence it satisfies an estimate

$$\left\| \sum_{k=1}^{\infty} k^{\frac{1}{2}} \varepsilon_k \otimes (T^{k-1}(x) - T^{k+1}(x)) \right\|_{\text{Rad}(L^p(M))} \lesssim \|x\|_{L^p}.$$

REFERENCES

- [1] W. Arendt, and S. Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. 240 (2002), 311-343.
- [2] C. Anantharaman-Delaroche, *On ergodic theorems for free group actions on noncommutative spaces*, Probab. Theory Relat. Fields 135 (2006), 520-546.
- [3] C. Arhancet, *On Matsaev's conjecture for contractions on noncommutative L_p -spaces*, J. Operator Theory, to appear (arXiv:1009.1292).
- [4] C. Arhancet, and C. Le Merdy, *Dilation of Ritt operators on L_p -spaces*, Preprint 2011, arXiv:1106.1513.
- [5] C. Badéa, B. Beckermann, and M. Crouzeix, *Intersections of several disks of the Riemann sphere as K -spectral sets*, Comm. Pure Appl. Anal. 8 (2009), 37-54.
- [6] E. Berkson, and T. A. Gillespie, *Spectral decompositions and harmonic analysis on UMD Banach spaces*, Studia Math. 112 (1994), 13-49.
- [7] S. Blunck, *Maximal regularity of discrete and continuous time evolution equations*, Studia Math. 146 (2001), no. 2, 157-176.
- [8] S. Blunck, *Analyticity and discrete maximal regularity on L_p -spaces*, J. Funct. Anal. 183 (2001), 211-230.
- [9] P. Clément, B. de Pagter, F. A. Sukochev, and H. Witvliet, *Schauder decompositions and multiplier theorems*, Studia Math. 138 (2000), 135-163.
- [10] M. Cowling, I. Doust, A. McIntosh, and A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, J. Aust. Math. Soc., Ser. A 60 (1996), 51-89.
- [11] M. Crouzeix, *Numerical range and functional calculus in Hilbert space*, J. Funct. Anal. 244 (2007), 668-690.
- [12] R. de Laubenfels, *Similarity to a contraction, for power-bounded operators with finite peripheral spectrum*, Trans. Amer. Math. Soc. 350 (1998), 3169-3191.
- [13] B. Delyon, and F. Delyon, *Generalization of von Neumann's spectral sets and integral representation of operators*, Bull. Soc. Math. France 127 (1999), 25-42.
- [14] H. R. Dowson, *Spectral theory of linear operators*, London Math. Soc. Monographs 12, Academic Press, London-New York, 1978. xii+422 pp.
- [15] E. Franks, and A. McIntosh, *Discrete quadratic estimates and holomorphic functional calculi in Banach spaces*, Bull. Austral. Math. Soc. 58 (1998), 271-290.
- [16] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, 96. Academic Press, New York-London, 1981. xvi+467 pp.
- [17] U. Haagerup, and M. Musat, *Factorization and dilation problems for completely positive maps on von Neumann algebras*, Commun. Math. Phys. 303 (2011), 555-594.
- [18] M. Haase, *Spectral mapping theorems for holomorphic functional calculi*, J. London Math. Soc. 71 (2005), 723-739.
- [19] M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, 169, Birkhäuser Verlag, Basel, 2006. xiv+392 pp.

- [20] M. Hieber, and J. Prüss, *Functional calculi for linear operators in vector-valued L^p -spaces via the transference principle*, Adv. Diff. Eq. 3 (1998), 847-872.
- [21] M. Junge, C. Le Merdy, and Q. Xu, *H^∞ -functional calculus and square functions on noncommutative L^p -spaces*, Soc. Math. France, Astérisque 305, 2006. vi+138 pp.
- [22] M. Junge, É. Ricard, and D. Shlyakhtenko, *Noncommutative diffusion semigroups and free probability*, in preparation.
- [23] N. J. Kalton, and P. Portal, *Remarks on ℓ_1 and ℓ_∞ -maximal regularity for power bounded operators*, J. Aust. Math. Soc. 84 (2008), 345-365.
- [24] N. J. Kalton, and L. Weis, *The H^∞ -calculus and sums of closed operators*, Math. Ann. 321 (2001), 319-345.
- [25] N. J. Kalton, and L. Weis, *The H^∞ -functional calculus and square function estimates*, Unpublished manuscript (2004).
- [26] C. Krieger, and C. Le Merdy, *Tensor extension properties of $C(K)$ -representations and applications to unconditionality*, J. Aust. Math. Soc. 88 (2010), 205-230.
- [27] P. C. Kunstmann, and L. Weis, *Maximal L_p -regularity for parabolic equations, Fourier multiplier theorems and H^∞ -functional calculus*, in “Functional analytic methods for evolution equations”, pp. 65-311, Lecture Notes in Math., 1855, Springer, Berlin, 2004.
- [28] S. Kwapien, *On Banach spaces containing c_0* , Studia Math. 52 (1974), 187-188.
- [29] C. Le Merdy, *H^∞ -functional calculus and applications to maximal regularity*, Publ. Math. Besançon 16 (1998), 41-77.
- [30] C. Le Merdy, *The similarity problem for bounded analytic semigroups on Hilbert spaces*, Semigroup Forum 56 (1998), 205-224.
- [31] C. Le Merdy, *The Weiss conjecture for bounded analytic semigroups*, J. London Math. Soc. 67 (2003), 715-738.
- [32] C. Le Merdy, *A sharp equivalence between H^∞ functional calculus and square function estimates*, Preprint 2011, arXiv:1111.3719.
- [33] C. Le Merdy, and Q. Xu, *Maximal theorems and square functions for analytic operators on L^p -spaces*, J. London Math. Soc., to appear (arXiv:1011.1360).
- [34] C. Le Merdy, and Q. Xu, *Strong q -variation inequalities for analytic semigroups*, Annales Inst. Fourier, to appear (arXiv:1103.2874).
- [35] J. Lindenstrauss, and L. Tzafriri, *Classical Banach spaces II*, Springer-Verlag, Berlin-New York, 1979. x+243 pp.
- [36] F. Lust-Piquard, *Inégalités de Khintchine dans C^p ($1 < p < \infty$)* (French), C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), 289-292.
- [37] F. Lust-Piquard, and G. Pisier, *Noncommutative Khintchine and Paley inequalities*, Ark. Mat. 29 (1991), 241-260.
- [38] Yu. Lyubich, *Spectral localization, power boundedness and invariant subspaces under Ritt's type condition*, Studia Math. 134 (1999), 153-167.
- [39] A. McIntosh, *Operators which have an H^∞ functional calculus*, Proc. CMA Canberra 14 (1986), 210-231.
- [40] B. Nagy, and J. Zemanek, *A resolvent condition implying power boundedness*, Studia Math. 134 (1999), 143-151.
- [41] O. Nevanlinna, *Convergence of iterations for linear equations*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993. viii+177 pp.
- [42] V. I. Paulsen, *Toward a theory of K -spectral sets*, in “Surveys of some recent results in operator theory, Vol. I”, pp. 221-240, Pitman Res. Notes Math. Ser., 171, Longman Sci. Tech., Harlow, 1988.
- [43] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002. xii+300 pp.
- [44] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Compos. Math. 37 (1978), 3-19.
- [45] G. Pisier, *Non-commutative vector valued L_p -spaces and completely p -summing maps*, Soc. Math. France, Astérisque 247, 1998. vi+131 pp.

- [46] G. Pisier, *Similarity problems and completely bounded maps* (Second, expanded edition), Lecture Notes in Mathematics, 1618 Springer-Verlag, Berlin, 2001. viii+198 pp.
- [47] G. Pisier, and Q. Xu, *Non-commutative martingale inequalities*, Comm. Math. Phys. 189 (1997), 667-698.
- [48] G. Pisier, and Q. Xu, *Non-commutative L^p -spaces*, pp. 1459-1517 in “Handbook of the Geometry of Banach Spaces”, Vol. II, edited by W.B. Johnson and J. Lindenstrauss, Elsevier, 2003.
- [49] É. Ricard, *A Markov dilation for self-adjoint Schur multipliers*, Proc. Amer. Math. Soc. 136 (2008), 4365-4372.
- [50] W. Rudin, *Real and complex analysis* (Third edition), McGraw-Hill Book Co., New York, 1987. xiv+416 pp.
- [51] E.M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. Math. Studies, Princeton, University Press, 1970.
- [52] P. Vitse, *A band limited and Besov class functional calculus for Tadmor-Ritt operators*, Arch. Math. (Basel) 85 (2005), 374-385.
- [53] P. Vitse, *A Besov class functional calculus for bounded holomorphic semigroups*, J. Funct. Anal. 228 (2005), 245-269.
- [54] L. Weis, *Operator valued Fourier multiplier theorems and maximal regularity*, Math. Ann. 319 (2001), 735-758.

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